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Journal references in Mathematical Reviews are now given in the following form: J. Brobdingnag. Acad. Sci. (7) **4 (82)** (1952/53), no. 3, 17–42 (1954), where after the abbreviated title one has: (series number) volume number (volume number in first series if given) (nominal date), issue number if necessary, first page–last page (imprint date). In case only one date is given, this will usually be interpreted as the nominal date and printed immediately after the volume number (this is a change from past practice in Mathematical Reviews where a single date has been interpreted as the imprint date). If no volume number is given, the year will be used in its place. The symbol ★ precedes the title of a book or other non-periodical which is being reviewed as a whole.

References to reviews in Mathematical Reviews before volume 20 (1959) are by volume and page number, as MR **19**, 532: from volume 20 on, by volume and review number, as MR **20** #4387. Reviews reprinted from Applied Mechanics Reviews, Referativnyĭ Žurnal, or Zentralblatt für Mathematik are identified in parentheses following the reviewer's name by AMR, RŽMat (or RŽMeh, RŽAstr. Geod.), Zbl, respectively.

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Mathematical Reviews

Vol. 22, No. 10A October, 1961 Reviews 9424–9980

GENERAL

See also B10219.

9424:

★Indian Institute of Science. Golden Jubilee Research Volume: 1909–1959. Indian Institute of Science, Bangalore, 1959. vii+411 pp.

Contributions by the staff and students of the Institute. Articles of mathematical interest will be reviewed individually.

9425:

Ledermann, Walter. ★Complex numbers. Library of Mathematics. Routledge and Kegan Paul, London, 1960. vi+62 pp. 5 s.

An elementary textbook designed for use in British sixth form or first year university work. Contains a clear exposition of algebraic and geometric properties of complex numbers, together with the first principles of the theory of functions of a complex variable.

9426:

Li, Wen-Hsiung. ★Engineering analysis. Prentice-Hall Engineering Science Series. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1960. viii+362 pp. $11.00.

This textbook is an attempt at presenting mathematics as a technical language and a practical tool to undergraduate students in engineering. The topics covered are differential calculus, vector algebra, applications of the integral calculus, ordinary differential equations, and Fourier series. In the opinion of the reviewer, the author has succeeded admirably in integrating the mathematical topics presented with the physical applications. By the author's judicious choice of physical problems to illustrate the underlying mathematical principles the student of engineering is motivated to study the various topics presented to a much greater degree than if he were studying the same mathematical topics presented in the conventional manner.

The author has taken his examples from hydrostatics, vibration theory, electricity and magnetism, heat conduction, electric-circuit theory, and dynamics. The subject-matter has been arranged so that it may be understood by students with only an elementary knowledge of the differential calculus and who are taking a course in the integral calculus concurrently.

L. A. Pipes (Los Angeles, Calif.)

9427:

Menger, Karl. Gulliver in the land without one, two, three. Math. Gaz. **43** (1959), 241–250.

In his campaign for more sensible mathematical notations, the author resorts to an amusing comparison with an equally nonsensical arithmetical notation found by Gulliver in one of his travels. Gulliver's islanders use the symbols |, |ᵖ, |ᵗ, 4|, 5|, ··· (read stix, stixpair, stixtrip, four stix, five stix, ···) for the natural numbers, and find themselves confronted with countless and comical notational difficulties, quite similar to those encountered by present-day teachers in the classroom (and also by most abstract mathematical writers in need of uniform and self-explanatory notations), when dealing with "functions" such as x, x^n, $\log x$, or with "values of functions for the value x of the variable", or with "functions of functions", and so on. In fairness, one must remark that many branches of mathematics (notably algebra, topology, algebraic geometry) do use, often since their inception, notations not dissimilar from those proposed by the author, with the possible exception of power functions; for the author's notations, see his *Calculus, a modern approach* [Ginn, New York, 1955; MR **17**, 351]. See also H. A. Pogorzelski, Amer. Math. Monthly **66** (1959), 885 [MR **21** #5549].

I. Barsotti (Providence, R.I.)

HISTORY AND BIOGRAPHY

9428:

Porcu, Livio. Il metodo di falso posizione. I, II, III. Period. Mat. (4) **38** (1960), 95–112, 149–154, 213–227.

L'auteur croit utile de rappeler l'ancienne méthode de la fausse position dont il souligne la grande valeur didactique. La première partie de l'article est consacrée à l'histoire de cette méthode qui, connue déjà des chinois, des hindous et des arabes, a trouvé son plein développement dans les œuvres de Léonard de Pise et surtout dans la "Summa" de Luca Pacioli. L'auteur expose l'essentiel de la méthode en faisant recours au texte original de la "Summa" et en y empruntant des exemples. Dans la seconde partie il expose les fondements algébriques et l'interprétation géométrique de la méthode. Dans la troisième partie il donne plusieurs exemples de son application aux diverses constructions géométriques et à la résolution approchée des équations.

Z. Opial (Kraków)

9429:

Agostini, Amedeo. La matematica dei Babilonesi. Period. Mat. (4) **37** (1959), 201–212.

Après une courte introduction historique l'auteur passe en revue les principales connaissances des babiloniens en mathématiques. Dans la partie arithmétique et géométrique il se contente d'énumérer les procédés et les

formules dont ils se servaient, dans la partie algébrique (les problèmes du second et du troisième dègré) il commente plusieurs textes originaux. L'article devait constituer le premier chapitre d'un ouvrage sur l'histoire des mathématiques dans l'Antiquité qui restera inachevé à cause de la mort de l'auteur. *Z. Opial* (Kraków)

9430:
In memoria di Amedeo Agostini. Period. Mat. (4) **37** (1959), 245–251.

9431:
★La disputa Leibniz-Newton sull'analisi. Scelta da documenti degli anni 1672–1716, traduzione e note di Gianfranco Cantelli. Enciclopedia di autori classici, 2. Editore Boringhieri, Torino, 1958. 241 pp. L. 1200.

The publication of documents on the Newton-Leibniz quarrel in a Library of Classical Authors is to be welcomed. Unfortunately, however, the situation is so complex that without the help of a scholarly prepared introduction it is impossible for the layman to form an impartial opinion on it. It is here where the book leaves much to be desired. The relatively few brief notes are not sufficient at all. Some of the sources for this translation are not very reliable, and new material has been brought to light during the past decades which should have been embodied in the explanatory notes. There is no index save for the headings of the 41 pieces and a list of the sources. Let us hope that the reader's interest will be stimulated enough to turn from this book to the originals; if he does so it will have served its purpose well. *C. J. Scriba* (Toronto)

9432:
Tintner, Gerhard. The statistical work of Oskar Anderson. J. Amer. Statist. Assoc. **56** (1961), 273–280.

Includes bibliography.

9433:
Aleksandrov, P. S.; Vishik, M. I.; Saul'ev, V. K.; El'sgol'ts, L. E. Lazar'Aronovich Lyusternik (on the occasion of his 60th birthday). Russian Math. Surveys **15** (1960), no. 2, 153–168.

Translation of Uspehi Mat. Nauk **15** (1960), no. 2 (92), 215–230 [MR **22** #2524].

9434:
Krein, M. G.; Shilov, G. E. Mark Aronovich Naimark (on the occasion of his fiftieth birthday). Russian Math. Surveys **15** (1960), no. 2, 169–174.

Translation of Uspehi Mat. Nauk **15** (1960), no. 2 (92), 231–236 [MR **22** #2525].

9435:
Galilei, Galileo. ★On motion & On mechanics. Comprising De Motu (ca. 1590), translated with introduction and notes by I. E. Drabkin; and Le Meccaniche (ca. 1600), translated with introduction and notes by Stillman Drake. Publications in Medieval Science, 5. The University of Wisconsin Press, Madison, Wis., 1960. ix + 193 pp. $5.00.

The English translation by I. E. Drabkin of Galileo's juvenile work on motion (written about 1590) is the first one to appear in a modern language. The Latin original was published by A. Favaro in Vol. I of the Edizione Nazionale (1890). Drabkin presents a translation of the text, which is considered by Favaro to be Galileo's first version of a treatise on motion; the text is preceded by an introduction and is accompanied by elaborate notes. The second part of the volume is occupied by an English translation by Stillman Drake of another early work of Galileo, the Italian treatise *Le Meccaniche* (written about 1600) and destined for use by his private pupils. Accordingly, it was not published during his lifetime. Only a French translation by Mersenne appeared in 1634. Hitherto it has not been obtainable in English. The translation now published is again elucidated by an introduction and a number of notes. *E. J. Dijksterhuis* (Bilthoven)

9436:
Turnbull, H. W. (Editor). ★The correspondence of Isaac Newton, Vol. II: 1676–1687. Published for the Royal Society. Cambridge University Press, New York, 1960. xiii + 552 pp. (7 plates) $25.00.

The period included by this volume includes that of writing and publishing the *Principia*. As in the previous volume, not only letters to and from Newton but also letters or extracts from other relevant correspondences are included, and everything possible is done to enable the reader to follow the circumstances and arguments.

The scene is well set by the first two letters. In the former Newton gives some of his views on alchemy, speaking of the "metallick particles" and "saline particles", which "enter and shake those bodies more fully & by their grossness shake ye dissolved particles more strongly then a subtiler agent would do." The second, from Leibniz, concerns the series for the sine and arcsine.

A good many valuable pieces are printed here for the first time. For example, the correspondence with Flamsteed, mainly on comets, is completed so as to form a real source work for mathematical astronomy. A side of Newton's character which has been somewhat veiled up to the present is now seen in pieces No. 217–221 and 225–226, which present the private aspects of his quarrel with Lucas, derivative from the earlier public controversy with Linus over refraction. They show that however desirous he was of avoiding differences, he knew how to pursue one hotly once into it: "Pray trouble your self no further to reconcile me wth truth but let us know your own mistakes."

No. 190 is dated 1664 by the editor and is regarded by him as giving "perhaps the earliest statement and proof of the fundamental theorem of the calculus." No. 191, dated 1665 by the editor, gives the binomial theorem for arbitrary rational exponent. No. 192 seems to be connected with the famous "Epistola posterior" sent to Oldenburg for Leibniz on 24 October 1676. These three pieces, not previously published, are printed along with the better known documents and letters pertaining to the sources of the later controversy over priority in the calculus. It is perhaps through scholarly principle that the notes from the *Commercium Epistolicum* of 1712 are reprinted in this connection, yet it is unfortunate that their polemic character is corrected only in the cases arising from demonstrable errors of transcription or interpretation.

The editor is not unfair to Leibniz in any way, yet the reprinting of comments from one side only casts a regrettable aura of bias on this part of the volume. However, from the documents themselves the reader easily forms the picture of Leibniz, whether earlier or later, standing alone in his thinking and dealing openly and elegantly, while Newton is put forward as their grudging champion by a school of British who scurry about behind the scenes deciding what is and what is not fit to be disclosed.

The other newly published pieces of main interest to mathematical readers are No. 271, which is an outline of a treatise on mathematics from 1684, and Nos. 311–316, consisting in a correspondence with G. Clerke regarding some difficulties in the *Principia*, just published. These difficulties concern neither the principles of mechanics nor any general method but rather details and terminology.

The volume, like its predecessor [Cambridge Univ. Press, New York, 1959; MR **21** #6313], is excellently edited and presented, but unfortunately it wants a preface to orient such readers as are not already at home in the mathematical scene of late seventeenth-century England. Also the editor does not reveal the grounds on which he chose to include certain selected fragments from the great mass of Newton's manuscript notes. The reader will be grateful for what he is given, but he may ask why the edition was planned in this unusual way, and when the rest of the non-epistolary manuscripts will be made available.

C. Truesdell (Basel)

9437:

Слуцкий, Е. Е. [Sluckiĭ, E. E.]. ★Избранные труды. Теория вероятностей, математическая статистика [Selected works. Theory of probability, mathematical statistics]. Izdat. Akad. Nauk SSSR, Moscow, 1960. 292 pp. (1 plate) 17.30 r.

Eighteen selected papers, with commentaries on several, and a detailed biography. *J. Wolfowitz* (Ithaca, N.Y.)

LOGIC AND FOUNDATIONS

9438:

Schütte, Kurt. ★Beweistheorie. Die Grundlehren der mathematischen Wissenschaften, Bd. 103. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1960. x+355 pp. DM 48.00.

The propositional and predicate calculi are treated in an unorthodox fashion, based upon the notions of positive and negative parts of a formula. (F is a positive part of F. If $\neg A$ is a positive (negative) part of F, then A is a negative (positive) part of F. If $A \vee B$ is a positive part of F, so are A and B. If $A \wedge B$ is a negative part of F, then so are A and B.) By $F[P_+]$ and $G[Q_-]$ we indicate that P is a positive part of F and Q is a negative part of G. A formula is simple if its minimal positive and negative parts are propositional variables. As axioms we take all simple formulas in which some propositional variable occurs both as a positive and negative part. The rules of inference are: (S1a) $F[A_+], F[B_+] \rightarrow F[(A \wedge B)_+]$; (S1b) $F[A_-], F[B_-] \rightarrow F[(A \vee B)_-]$. For the predicate calculus, the axioms are the formulas in which some atomic formula occurs both as a positive and negative part. The rules of inference are (S1a), (S1b), and (S2a) $F[A(a)_+] \rightarrow F[\wedge x\, A(x)_+]$, (S2b) $F[A(a)_-] \rightarrow F[\vee x\, A(x)_-]$, (S3a) $F[\vee x\, A(x)_+] \vee A(a) \rightarrow F[\vee x\, A(x)_+]$, (S3b) $F[\wedge x\, A(x)_-] \vee \neg A(a) \rightarrow F[\wedge x\, A(x)_-]$. A completeness proof is given in the spirit of Hintikka [Acta Philos. Fenn. 8 (1955), 7–55; MR **16**, 1079] and Beth's work on semantic tableaux.

For use in proofs by transfinite induction, a particular recursive ordinal is defined by means of a well-ordering $\prec$ of the natural numbers. Let $(b)_i$ be the highest power of the ith prime ($i=0, 1, 2, \cdots$) dividing b. Now (defining at the same time another well-ordering $\subset$): (R1) If $b \neq 0$, then $0 \subset b$ and $0 \prec b$. (R2) If $a \neq 0$, $b \neq 0$, and $(a)_i \prec (b)_i$ while $(a)_j = (b)_j$ for all $j > i$, then $a \subset b$. (R3) If $a \neq 0$, $b \neq 0$, $a \subset b$ and $(a)_i \prec b$ for all i, then $a \prec b$. (R4) If $a \neq 0$, $b \neq 0$, $b \subset a$ and $a \prec (b)_i$ or $a = (b)_i$ for at least one i, then $a \prec b$. Some ordinal arithmetic is developed in a somewhat more general setting. Elementary formal number theory is presented by means of a system Ψ which essentially has the same strength as the standard systems for first-order number theory based on Peano's postulates. (However, the formal character is partially destroyed by admission of predicates and functors corresponding to arbitrary effectively calculable predicates and functions and by inclusion among the axioms of certain formulas which can be constructively shown to be verifiable. The notions of "effectively calculable" and "constructive" are left at the intuitive level, although they are used throughout the book.) The consistency of Ψ is shown by attaching ordinal numbers to proof-trees and then by proving that every provable closed formulas without quantifiers is true. A semi-formal "constructively complete" system Ω is introduced on the basis of a constructivistic form of a rule of infinite induction. Again by attaching ordinals to proof-trees, cuts are shown to be eliminable, and, from this, it follows easily that Ω is a consistent extension of Ψ. Transfinite induction based upon $\prec$ is provable in Ψ up to but not including the first ε-number. It follows that Ψ is ω-incomplete. However, transfinite induction is provable in Ω for every (representable) ordinal. (It is indicated that these results hold for any recursive ordinal greater than that defined by $\prec$.)

A type-free semi-formal higher-order logic Λ is developed without any restriction except that the propositional calculus is not present in its full strength. The semi-formal character of Λ results from the (surprising) inclusion of an infinite rule of induction (on all constant terms). In contrast to Λ, a semi-formal ramified type theory T (including a rule of infinite induction) is studied. Also considered are two restricted forms Γ and Γ^* of T suitable for real-number theory (since they contain only those predicate variables with arguments of type 0). All the systems Λ, T, Γ, Γ^* are proved consistent. The ι-operator for definite descriptions can be consistently added to the type-free system Λ. A choice operator exists for Λ by virtue of a recursive enumeration of the constant terms. Similar results hold for T. Real analysis is founded upon a special type-free system M. The theory of natural numbers is included in M, and, by a suitable mapping of pairs of integers into integers, the theory of rationals can be developed as a part of the theory of natural numbers. Then real numbers are defined as predicates which define the lower open half of Dedekind cuts. The theory of real sequences, functions, and point-sets is sketched, and, finally, the merits and demerits of type-free analysis as opposed to a (ramified) type-theoretic analysis are discussed. *E. Mendelson* (New York)

9439:

Ádám, András. Zur Theorie der Wahrheitsfunktionen. Acta Sci. Math. Szeged **21** (1960), 47–52.

This paper is related to the problem of simplifying truth-functions. Let f_1, f_2, f_3 be Boolean functions, $x_1, x_2, \cdots$ distinct Boolean variables (assuming only the values T or F). Then f_1 is a simple superposition without repetition of variables, of f_2 on f_3, if it can be written $f_1(x_1, \cdots, x_n) = f_2(x_1, \cdots, x_k, f_3(x_{k+1}, \cdots, x_n))$. There are three theorems concerning such superpositions.

L. N. Gál (Minneapolis, Minn.)

9440:

Žuravlev, Yu. I. On the impossibility of constructing minimal disjunctive normal forms for functions of the algebra of logic in a single class of algorithms. Dokl. Akad. Nauk SSSR **132** (1960), 504–506 (Russian); translated as Soviet Math. Dokl. **1**, 581–583.

It is shown that certain simple kinds of algorithms, for finding those conjunctions in a disjunctive normal form which do not occur in any corresponding minimal disjunctive normal form, do not exist.

E. Mendelson (New York)

9441a:

Ohnishi, Masao; Matsumoto, Kazuo. Gentzen method in modal calculi. Osaka Math. J. **9** (1957), 113–130.

9441b:

Ohnishi, Masao; Matsumoto, Kazuo. Gentzen method in modal calculi. II. Osaka Math. J. **11** (1959), 115–120.

Der Gentzen-Kalkül LK wird durch die Modaloperatoren $\Box$ (notwendig) und $\Diamond$ (möglich) erweitert mit Hilfe folgender Regeln

$$\frac{\Gamma \to \Theta, \alpha}{\Gamma \to \Theta, \Diamond\alpha}, \qquad \frac{\alpha, \Theta \to \Gamma}{\Box\alpha, \Theta \to \Gamma}$$

und (wahlweise) mit

(S5*) $$\frac{\alpha, \Diamond\Gamma \to \Diamond\Theta}{\Diamond\alpha, \Diamond\Gamma \to \Diamond\Theta}, \qquad \frac{\Box\Theta \to \Box\Gamma, \alpha,}{\Box\Theta \to \Box\Gamma, \Box\alpha},$$

(S4*) $$\frac{\alpha \to \Diamond\Theta}{\Diamond\alpha \to \Diamond\Theta}, \qquad \frac{\Box\Theta \to \alpha}{\Box\Theta \to \Box\alpha},$$

(M*) $$\frac{\alpha \to \Theta}{\Diamond\alpha \to \Diamond\Theta}, \qquad \frac{\Theta \to \alpha}{\Box\Theta \to \Box\alpha}$$

Diese modalen Sequenzenkalküle sind äquivalent mit den Modalkalkülen S5, S4 und M von Lewis-Langford und v. Wright. Verff. beweisen in der ersten Arbeit den Gentzenschen Hauptsatz für S4* und M*. Dadurch ergeben sich neue Entscheidungsverfahren für die Modalkalküle S4 und M.

Die zweite Arbeit zeigt, dass der Hauptsatz nicht für S5* gilt. Verff. geben einen Beweis für das Parrysche Resultat, dass jede Formel von S5* äquivalent ist zu einer Formel, die höchstens den Grad 1 (bzgl. $\Box$, $\Diamond$) hat und beweisen dann den Hauptsatz für den Sequenzenkalkül S5**, der durch Einschränkung aller Regeln von S5* auf Formeln höchstens vom Grad 1 entsteht. So erhält man auch ein neues Entscheidungsverfahren für S5.

P. Lorenzen (Kiel)

9442a:

Matsumoto, K.; Ohnishi, M.; Ridder, J. Die Gentzenschen Schluszverfahren in modalen Aussagenlogiken. I. Nederl. Akad. Wetensch. Proc. Ser. A **60** = Indag. Math. **19** (1957), 481–491.

9442b:

Ridder, J. Die Gentzenschen Schluszverfahren in modalen Aussagenlogiken. II, III. Nederl. Akad. Wetensch. Proc. Ser. A **61** = Indag. Math. **20** (1958), 16–27.

Diese Arbeiten bringen Ergänzungen zu den Arbeiten von J. Ridder [dieselben Proc. **55** (1952), 213–223, 459–467; **56** (1953), 1–11, 99–110; **57** (1954), 117–128, 389–396; MR **14**, 527, 1052; **15**, 90; **16**, 2; **17**, 224]. Aufgrund eines Beispiels der beiden ersten Autoren muss nämlich die Formulierung des (Gentzenschen) Hauptsatzes für die betrachteten modalen Sequenzenkalküle durch Hinzunahme zulässiger Regeln modifiziert werden. Nur für die Sequenzenkalküle, die dem Lewisschen Kalkül S5 entsprechen, lässt sich der Hauptsatz nicht aufrecht erhalten.

P. Lorenzen (Kiel)

9443:

Kleene, S. C. Extension of an effectively generated class of functions by enumeration. Colloq. Math. **6** (1958), 67–78.

If r is the Gödel number of a primitive recursive (p.r.) description of a one-place function φ in the function θ, set $[r]^\theta = \varphi$, and otherwise set $[r]^\theta = \lambda x \cdot 0$; and write $[r] = [r]^{\lambda x \cdot 0}$. The set O' of p.r. ordinal notations and the relation $<_{O'}$ are defined exactly like O and $<_O$, except that $\{r\}$ is replaced by $[r]$ in the conditions for $3 \cdot 5^r \in O'$ and $a <_{O'} 3 \cdot 5^r$; i.e., limit ordinals are represented by p.r. fundamental sequences. It is proved that the ordinals represented in O' are the same as those represented in O.

The main result of the paper is the introduction of a hierarchy $\{C_d | d \in O'\}$ of general recursive functions, where C_d is the class of all functions φ which are p.r. in ρ_d (written $\varphi \dot{\leqq} \rho_d$), and ρ_d is associated with $d \in O'$ as follows: $\rho_1 = \lambda x \cdot O$, $\rho_{2^a}(a) = [(a)_0]^{\rho_a}((a)_1)$, and $\rho_{3 \cdot 5^a}(a) = \rho_{[a]((a)_0)}((a)_1)$. A number of problems concerning this hierarchy are listed, the most interesting of which are now solved, though none of the solutions is yet in print. E.g., S. Feferman has shown that $\bigcup_{|d| \leqq \omega} C_d$ is the set of all recursive functions, from which it also follows that C_d and C_e need not coincide for $|d| = |e| \geqq \omega^2$. On the other hand, P. Axt [Trans. Amer. Math. Soc. **92** (1959), 85–105] has shown that $C_d = C_e$ for $|d| = |e| < \omega^2$. Axt has also proved that the 3-fold (nested) recursive functions are in $\bigcup_{|d| < \omega^2} C_d$.

In analogy with the theory of recursive degrees of unsolvability, a theory of p.r. degrees, determined by the equivalence $\varphi \dot{\leqq} \psi \wedge \psi \dot{\leqq} \varphi$, is outlined. The reader is referred by the author to Axt's dissertation for the details as to how far the analogy holds.

Author's corrections (Oct. 13, 1959): P. 68, line 17, for "$\varphi_{2b}(a)$" read "$\varphi_{2b}(2a)$"; p. 68, first line of footnote should read "(Added December 17, 1957.) The"; p. 68, 5th line of footnote, last inferior numeral zero should be inferior capital italic "oh"; p. 69, line 3 from bottom of text, for "β" read "Θ"; p. 70, line 4, for "$\langle 0, m, i \rangle$" read "$\langle 0, m_i, i \rangle$"; p. 70, first line of 4, delete "a"; p. 70,

footnote (6) next to bottom line, for "θ" read "Ψ"; p. 74, in (15), for "$h(y, a, b)$" read "$h(y, b, a)$"; p. 75, line 8, for "y^y" read "y_n"; p. 76, line 11, for "$< o'$" read "$< o''$"; p. 76, line 12, for "$(c_2$" read "$(c)_2$"; page 76, in the big display, last equation, for "$U((n_{1\ln(n)-1}$" read "$U((n)_{1\ln(n)-1}$"; p. 76, line 17 from bottom, for "1's" read "1's"; p. 77, top of display, for "l" read "1"; p. 77, 3rd line of **P 237** and top line, p. 78, the inferior "O" should be inferior to the inferior; p. 77, 3rd line of **P 237**, for "y,s" read "y's".

W. W. Tait (Stanford, Calif.)

9444:

Kreisel, G. Analysis of the Cantor-Bendixson theorem by means of the analytic hierarchy. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **7** (1959), 621–626. (Russian summary, unbound insert)

Verf. gibt zunächst einen vereinfachten Beweis einer "konstruktiven" Verschärfung des Cantor-Bendixson-Theorems (C-B). F sei eine abgeschlossene Menge reeller Zahlen aus [0, 1]. Das Komplement in [0, 1] ist die Vereinigung einer Menge G offener rationaler Intervalle. Da die rationalen Zahlen abzählbar sind, ist G durch eine Menge natürlicher Zahlen darstellbar. Gehört diese Menge in der Kleene-Mostowski-Hierarchie zu $\Pi_n{}^1$ bzw. $\Sigma_n{}^1$, so heisst auch F zu $\Pi_n{}^1$ bzw. $\Sigma_n{}^1$ gehörig. (C-B): Wenn F zu $\Pi_1{}^1$ gehört, so auch der perfekte Kern F_p und der Rest $F-F_p$ kann durch eine in $\Pi_1{}^1$ rekursive Funktion abgezählt werden. Verf. zeigt auch, dass diese Resultate optimal sind: es gibt ein rekursives (d.h. zu $\Pi_0{}^0 \cap \Sigma_0{}^0$ gehöriges) F, sodass weder F_p noch eine Abzählung von $F-F_p$ zu $\Sigma_1{}^1$ gehören.

P. Lorenzen (Kiel)

9445:

Beth, Evert W. ★The foundations of mathematics: A study in the philosophy of science. Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Company, Amsterdam, 1959. xxvi+741 pp. $13.25.

The author presents a wide-ranging survey of philosophical problems and technical results on the foundations of mathematics. Part I is historical, starting with ancient Greek philosophy and ending with contemporary metamathematics. Like a good deal of the rest of the book, it is too much a sequence of commentaries rather than a complete exposition, and would not be easily understood by a novice in the field. Part II, "Elementary axiomatics", treats of various unformalized mathematical theories. Beginning with Peano's postulates, the author constructs the rationals and reals, and even includes an exhaustive treatment of p-adic extensions of the rationals. After a brief discussion of abstract algebra, there are fairly detailed presentations of the axiomatics of elementary Euclidean geometry, topological spaces, and Boolean algebras.

Not until part III, "Formalised axiomatics", and part IV, "Non-elementary metamathematics", do we enter into logic proper, with an exposition of topics in sentential logic, quantification theory, formalized number theory, recursive functions, arithmetization of syntax, theory of models, etc. The author's own approach to elementary logic by means of semantic tableaux is emphasized, but the scattering of the explanation of this interesting method over so many different sections makes it rather laborious to master. Part V, "The existence of mathematical entities", is divided into chapters on Logicism (Frege, Russell), Cantorism (naïve and axiomatic set theory), Intuitionism, and Nominalism. The chapter on Intuitionism is full of interesting material. The exposition of axiomatic set theory includes, among many other things, an outline of the relative consistency proof for the axiom of choice and the generalized continuum hypothesis. In the lucid part VI, "The Paradoxes", various solutions (axiomatic set theory, theory of types, etc.) are examined. Included are analyses of the semantic paradoxes and of the "paradox" of Skolem-Löwenheim. Part VII, "Application of set theory and topology in metamathematics", gives perhaps the first connected exposition of the connections between descriptive set theory (Borel sets, analytic sets, etc.) and mathematical logic. In part VIII, "Recursive functions, sets, and predicates", the theorems of Gödel and Rosser, as well as results on undecidability, are derived. Unfortunately, Turing machines are not mentioned, nor is there any space devoted to the justification of Church's Thesis. In his "Concluding remarks", part IX, on "Metamathematics, philosophy of mathematics, and general philosophy", the author displays a reasonable, even-tempered spirit.

This review cannot convey the vast amount of material present in this book. Although no one subject is treated in sufficient detail, there is enough to kindle the reader's interest and make him dip into the literature himself. On the debit side, there is little indication given to the beginner which of the many proofs omitted are difficult and which are easy. However, there are helpful exercises scattered through the book, and, as an appendix, a collection of exercises, many of them containing significant extensions of the material of the text.

E. Mendelson (New York)

9446:

Hintikka, K. Jaakko J. A new approach to sentential logic. Soc. Sci. Fenn. Comment. Phys.-Math. **17** (1953), no. 2, 14 pp.

Das Wesentliche des neuen Ansatzes des Verf. ist inzwischen schon in die Lehrbücher übergegangen. In Hilbert und Ackerman, *Grundzüge der theoretischen Logik* [4te Aufl., Springer, Berlin, 1959; MR **21** #3316], findet sich z.B. für die aus Aussagevariablen A, B, $\cdots$ allein mit $\vee$, $\bar{}$ zusammengesetzten Formeln K, L, $\cdots$ folgendes System (der Einfachheit halber sei hier für $\vee$ Kommutativität und Assoziativität vorausgesetzt). Grundformeln: $K \vee A \vee \bar{A}$ wenn K nur aus (eventuell negierten) Aussagevariablen mit $\vee$ allein zusammengesetzt ist. Regeln: (a) $K \vee L \Rightarrow K \vee \bar{\bar{L}}$; (b) $K \vee L_1$, $K \vee \bar{L}_2 \Rightarrow K \vee \overline{L_1 \vee L_2}$. — Zur Entscheidung über die Ableitbarkeit einer Formel verfolgt man — von der Formel selbst ausgehend — rückwärts alle möglichen Schritte nach (a) und (b). Der Gedanke des Verf., der gleichzeitig und unabhängig auch von Beth [siehe #9445] gefunden wurde, besteht darin, für die Rückwärtsschritte eine semantische Deutung zu geben — und diese dann, unabhängig von allen Kalkülen, an den Anfang zu stellen. Die Rückwärtsschritte erscheinen so als Schritte auf der Suche nach einer falsifizierenden Belegung einer gegebenen Formel.

Verf. führt diesen Gedanken am dualen Fall durch: er sucht nach einer verifizierenden Belegung für Formeln, die nur mit $\wedge$, $\bar{}$ zusammengesetzt sind, und erhält also folgende Schritte: (a') Wenn $K \wedge \bar{\bar{L}}$ verifizierbar ist, dann

ist $K \wedge L$ verifizierbar; (b') Wenn $K \wedge \overline{L_1 \wedge L_2}$ verifizierbar ist, dann ist $K \wedge \bar{L}_1$ oder $K \wedge \bar{L}_2$ verifizierbar.

Gleichzeitig verallgemeinert Verf. seine Resultate auf unendliche Konjunktionen. Die Übertragung dieser Methode der "semantischen Tableaus", wie sie nach Beth heisst, auf die Quantorenlogik hat Verf. in einer späteren Arbeit [Soc. Sci. Fenn. Comment. Phys.-Math. **17** (1955), no. 12; MR **22** #4633] durchgeführt.

P. Lorenzen (Kiel)

9447:

Cassina, Ugo. ★Critica dei principî della matematica e questioni di logica. Edizioni Cremonese, Rome, 1961. vii+519 pp. L. 4500.

A collection mainly of expository papers, published between 1928 and 1956. Among the subjects treated are: the notions of length and area; elements of set theory; Peano's formal system; axioms of Euclidean geometry.

A. Heyting (Amsterdam)

9448:

Mal'cev, A. I. Some correspondences between rings and groups. Mat. Sb. (N.S.) **50 (92)** (1960), 257–266. (Russian)

The elementary theory of (not necessarily associative) rings and the elementary theory of a certain class of metabelian groups are shown to be relatively interpretable in one another. From the undecidability of the theory of rings it then follows that various classes of groups have undecidable theories; among others, the metabelian groups, free metabelian groups of rank greater than one, and metabelian groups with the relation $x^p = 1$, where p is an odd prime.

E. Mendelson (New York)

9449:

Yablonskiĭ, S. V. On limit logics. Dokl. Akad. Nauk SSSR **118** (1958), 657–660. (Russian)

The author understands by "logic" a set of functions of any number of arguments ranging over a certain finite or infinite set and taking values in that set. For such logics the author defines homomorphism, isomorphism, etc., in a sense agreeing in principle with usual practice. The logic P_k is that formed by all functions of the indicated sort relative to the set $(0, 1, \cdots, k-1)$; $P_{\aleph_0}$ that similarly formed relative to the set of all natural numbers. Evidently $P_{\aleph_0}$ contains a continuum of functions. The author calls "limit logic" (predel'naya logika) a subset $\mathscr{P}$ of $P_{\aleph_0}$ such that (1) $\mathscr{P}$ consists of denumerably many functions, (2) $\mathscr{P}$ is closed under combinations of functions by substitution, (3) $\mathscr{P}$ contains a homomorphic image of every P_k for $k \geq 2$. The author's thesis is that there is a continuum of limit logics no two of which are isomorphic to one another.

H. B. Curry (University Park, Pa.)

SET THEORY

See also 9959.

9450:

Sierpiński, W. On a certain plane set. Wiadom. Mat. (2) **3**, 123–125 (1959). (Polish)

The following theorem is proved. If the cardinality of a plane set Z is $< \aleph_0$ on every line parallel to the y-axis, then $k < \aleph_0$, where k is the cardinality of the set of lines parallel to the x-axis on each of which the cardinality of the complement CZ of Z is $\leq \aleph_0$. Problem: Are there three straight lines P_1, P_2, P_3 and a partition of the plane R_2 into three sets Z_1, Z_2, Z_3 such that the cardinality of Z_i is $< \aleph_0$ on every line parallel to P_i $(i = 1, 2, 3)$? [Cf. Sierpiński, C. R. Acad. Sci. Paris **232** (1951), 1046–1047; Bull. Soc. Roy. Sci. Liège **20** (1951), 297–299; MR **13**, 330.]

Đ. Kurepa (Zagreb)

9451:

Ionescu-Bujor, Constantin. Sur les transformations correspondantes par rapport à une transformation donnée. Bul. Inst. Politehn. București **21** (1959), no. 1, 13–27. (Russian, English and German summaries)

Pour deux transformations T, τ, la τ-transformée de T est, para définition, $\tau T \tau^{-1}$; pour deux ensembles de transformations $[T]$, $[\tau]$, le transformé de $[T]$ par $[\tau]$ est l'ensemble des $\tau T \tau^{-1}$ $(T \in [T], \tau \in [\tau])$. T est dit symétrique [resp. parfaitement symétrique] par rapport à $[\tau]$ si, pour chaque $\tau \in [\tau]$, le τ-transformé de T fait partie de $[T]$ [resp. coincide avec $[T]$]. Si E, E' sont deux ensembles de transformations chacun [parfaitement] symétrique par rapport à $[\tau]$, l'ensemble EE' des xx' $(x \in E, x' \in E')$ l'est aussi. L'auteur donne encore quelques autres propriétés élémentaires concernant les notions précédentes.

Đ. Kurepa (Zagreb)

9452:

Ionescu-Bujor, Constantin. Sur les suites de points obtenues par une suite de transformations. Bul. Inst. Politehn. București **20** (1958), no. 3, 13–22. (Russian, English and German summaries)

$1, T_1, T_2, \cdots$ étant une suite de transformations, elle engendre à partir d'une paire ordonnée de points M, N deux suites bien déterminées de points $M, M_1, M_2, \cdots$ et $N, N_1, N_2, \cdots$. On examine quelques propriétés élémentaires de telles suites (par exemple, on examine si elles ont de termes communs, si $N_p = \tau M_p$ pour une certaine transformation, etc.).

Đ. Kurepa (Zagreb)

COMBINATORIAL ANALYSIS

See also B9993.

9453:

Lafon-Augé. Construction d'une géométrie projective finie sur une algèbre. J. Math. Pures Appl. (9) **39** (1960), 197–200.

Using an idea of Archbold and Johnson [Ann. Math. Statist. **27** (1956), 624–632; MR **18**, 244], the author studies a linear associative algebra $\mathscr{A}$ of order 3 over J_2, the field of integers (mod 2). This algebra has eight elements, $\sum_{i=1}^{3} c_i e_i$ (each $c_i \in J_2$), with the following commutative multiplication: e_1 is a multiplicative identity, $e_2^2 = e_3$, and $0 = e_2 e_3 = e_3 e_2 = e_3^2$. Certain ordered triples (x_1, x_2, x_3) are taken as "points" and certain equations $\sum_{i=1}^{3} a_i x_i = 0$ define "lines" $(x_i \in \mathscr{A}$ and $a_i \in \mathscr{A}$; $i = 1, 2, 3)$. The point (x_1^*, x_2^*, x_3^*) lies on the line $\sum_{i=1}^{3} a_i x_i = 0$ if and only if $\sum_{i=1}^{3} a_i x_i^* = 0$. The resulting "geometry" has 112 points and 112 lines, with 12 points per line and 12

lines through each point. Incidence relations are such that a partially balanced incomplete block design is obtained if we identify points with varieties, lines with blocks, and points-on-lines with plots-in-blocks. Parameters for the design are $v=b=112$; $r=k=12$; $\lambda_1=1$, $\lambda_2=2$, $\lambda_3=4$; $n_1=96$, $n_2=12$, $n_3=3$; the numbers λ_i and n_i determining classes of associated varieties in the standard way.

{The reviewer would prefer to delete the word "projective" from the title, because the "geometry" here constructed is not a projective geometry in the usual sense.} *W. A. Pierce* (Syracuse, N.Y.)

9454:

Ore, Oystein. Note on Hamilton circuits. Amer. Math. Monthly **67** (1960), 55.

This note gives a sufficient condition for a graph to have a Hamiltonian polygon. *W. T. Tutte* (Toronto)

9455:

Nash-Williams, C. St. J. A. On orientations, connectivity and odd-vertex-pairings in finite graphs. Canad. J. Math. **12** (1960), 555–567.

Let U be a finite unoriented graph without loops. Let V be its set of vertices. A subset S of V is said to separate the vertices ξ and η if it contains one but not the other. The author defines the degree of $S \subseteq V$ as the number of edges of U with one end, but not both ends, in S. The connectivity $c(\xi, \eta)$ of ξ and η is defined as the minimum of the degrees of the subsets S of V separating ξ and η.

Now let an orientation be assigned to U. Then $x(S)$ is defined as the number of exits from S, that is the number of edges directed from a member of S to a member of $V-S$. The minimum of $x(S)$ for sets S containing ξ but not η is denoted by $a(\xi, \eta)$. The orientation is called admissible if $a(\xi, \eta) \geq [\frac{1}{2}c(\xi, \eta)]$ for each ordered pair $\{\xi, \eta\}$ of distinct vertices of U. The author proves that every finite unoriented graph has an admissible orientation.

W. T. Tutte (Toronto)

9456:

Foulkes, J. D. Directed graphs and assembly schedules. Proc. Sympos. Appl. Math., Vol. 10, pp. 281–289. American Mathematical Society, Providence, R.I., 1960.

Using terminology different from that of the author, (1) a "tournament" T is a complete oriented graph, (2) T is "strongly connected" or "strong" if any two points are mutually reachable by directed paths, (3) a "strong component" of T is a maximal strong subgraph, and (4) a "complete cycle" or "complete path" of T contains all its points. Redei has shown that every tournament has a complete path, in fact an odd number. The results obtained in this paper are that (a) the number of complete paths in T is the product of the number of complete paths in each strong component of T, and (b) T is strong if and only if it has a complete cycle. Among possible applications are: (i) assembly line problems; (ii) the problem of placing examination and other candidates in an order which maximally agrees with an ordering of subsets drawn up by several judges; (iii) the problem of analyzing a group of consumer preference tests which have been conducted on an AB basis.

F. Harary (Ann Arbor, Mich.)

ORDER, LATTICES

See also 9439, 9440, 9952, B9982.

9457:

Dolcher, Mario. Una proprietà delle famiglie numerabili di catene sopra un albero. Ann. Univ. Ferrara. Sez. VII (N.S.) **8** (1958/59), 7–16. (French summary)

Let T be a tree, i.e., any ordered set such that for every $x \in T$ the set $T(\cdot, x)$ of predecessors of x is well ordered. If T has a single initial member and if every member m of T has just r immediate followers and just one immediate predecessor (provided m is not the first element of T), T is called an r-tree (r being a given cardinal number). Theorem: If r is finite, let T, T' be any ordered pair of r-trees; let F, F' respectively be any countable system of maximal chains exhausting T, T' respectively; then there is an isomorphism $i: T \to T'$ carrying F onto F'.

Đ. Kurepa (Zagreb)

9458:

Finkbeiner, D. T. A semimodular imbedding of lattices. Canad. J. Math. **12** (1960), 582–591.

Sharpening results of Dilworth [Duke Math. J. **8** (1941), 286–299; MR **3**, 100], it is proved that any finite-dimensional lattice $\mathfrak{L}$ is isometrically isomorphic to a sublattice of a semimodular lattice $\mathfrak{S}$ which has the same number of points as $\mathfrak{L}$ and preserved basic properties of the join-irreducible arithmetic of $\mathfrak{L}$. This is shown by defining a pseudo-rank function s on $\mathfrak{L}$, and imbedding $\mathfrak{L}$ in the lattice $\mathfrak{M}$ constructed by introducing between each join-irreducible $q \in \mathfrak{L}$ and the element c which it covers a chain of $s(q)-s(c)-1$ elements. Q denotes the set of all join-irreducible elements of $\mathfrak{M}$ other than 0. The subsets of Q which are closed relative to a certain relation form the desired $\mathfrak{S}$. The question of minimality of the imbedding is discussed to some extent.

P. M. Whitman (Silver Spring, Md.)

9459:

Tamaschke, Olaf. Submodulare Verbände. Math. Z. **74** (1960), 186–190.

In der Arbeit wird eine Klasse längenendlicher Verbände betrachtet, welche die folgenden abgeschwächten Halbmodularitätsbedingungen erfüllen: ($\underline{S}$) Ist $x \cup y/y$ ein Primquotient, dann sind alle Primquotienten $p/q \subseteq x/x \cap y$ projektiv zu $x \cup y/y$; ($\bar{S}$) ist dual zu ($\underline{S}$). Ein längenendlicher Verband, der diese beide Bedingungen ($\underline{S}$), ($\bar{S}$) erfüllt, heisst submodular. Es wird gezeigt, dass submodulare Verbände durch ihre Kongruenzrelationen einfach gekennzeichnet werden können. Sei K die Menge aller Klassen zueinander projektiver Primquotienten in einem längenendlichen Verband L. L ist genau dann submodular, wenn es zu jeder Teilmenge $B \subseteq K$ eine Kongruenzrelation κ_B in L gibt, die genau die Klassen von B annulliert. Weitere Ergebnisse: Sind a, b Elemente eines submodularen Verbandes L, R_1 eine maximale Kette von a bis $a \cap b$ und R_2 eine maximale Kette von $a \cup b$ bis b, so ist jeder Primquotient der Kette R_i projektiv zu einem Primquotienten der Kette R_j ($i \neq j$; $i, j = 1, 2$). Ein Element n eines submodularen Verbandes L ist genau dann neutral, wenn eine der zwei äquivalenten Bedingungen erfüllt wird: (1) keine zwei Primquotienten p/q, r/s in L mit $q \geq n \geq r$ sind zueinander projektiv; (2) die Abbildung $x \to x \cap n$ ist ein Endomorphismus von L. Ein

Teilverband eines submodularen Verbandes ist im allgemeinen kein submodularer Verband; nicht jeder submodulare Verband erfüllt die Jordan-Dedekindsche Kettenbedingung. Dagegen ist jedes homomorphe Bild eines submodularen Verbandes wieder ein submodularer Verband. *V. Vilhelm* (Prague)

9460:

Murata, Kentaro. Additive ideal theory in multiplicative systems. J. Inst. Polytech. Osaka City Univ. Ser. A **10** (1959), 91–115.

In this paper the author attempts to obtain for multiplicative lattices decomposition results similar to the results obtained for rings and groups by C. W. Curtis [Amer. J. Math. **74** (1952), 687–700; MR **14**, 127] and E. Schenkman [Proc. Amer. Math. Soc. **9** (1958), 375–381; MR **20** #60]. Unfortunately, theorem 1 of this paper is not correct and a number of other results depend on this theorem. The difficulties in the proof of this theorem are contained in the sentence: "It is easy to see that the element $(a \cup u) \cap (a \cup (ev)^{(\omega)})$ is the greatest element of $j(a \cup u) \wedge K$." A counter-example to this theorem is given by the first example in a paper by M. Ward and R. P. Dilworth [Trans. Amer. Math. Soc. **45** (1939), 335–354]. *J. Hartmanis* (Schenectady, N.Y.)

9461:

Dilworth, R. P.; Crawley, Peter. Decomposition theory for lattices without chain conditions. Trans. Amer. Math. Soc. **96** (1960), 1–22.

An element c of a lattice L is called compact if whenever c is contained in the join of a set S of elements then c is contained in the join of a finite subset of S. A complete lattice L is called compactly generated if its compact elements generate L under unrestricted joins. In a complete lattice L, every element is compact if and only if L satisfies the ascending chain condition.

The decomposition theory of lattices is studied, generalizing work of Dilworth [Ann. of Math. (2) **41** (1940), 771–777; Duke Math. J. **8** (1941), 286–299; MR **2**, 120; **3**, 100]; the ascending chain condition is replaced by compact generation and the descending chain condition by atomicity. Thus the structure of the quotient lattice generated by elements covering a, though no longer finite-dimensional, is still related to the decompositions of a (now not necessarily finite). For instance, let a be an element of a semimodular, compactly generated, atomic lattice and let R be a set of completely meet-irreducible elements containing a. Then R can be extended to an irredundant decomposition of a if and only if R is irredundant over a. Some theorems about replacement of elements in two irredundant decompositions of an element of a modular lattice likewise generalize, but not so simply. An example shows that in a compactly generated atomic modular lattice an element can have two irredundant decompositions with different cardinalities. An atomic lattice L is called locally distributive (or locally modular) if u_a/a is distributive (or modular, respectively) for each $a \in L$, where u_a is the join of the elements covering a. A compactly generated atomic lattice L has unique irreducible decompositions if and only if it is locally distributive. A semimodular, compactly generated, atomic lattice has replaceable decompositions if and only if it is locally modular. *P. M. Whitman* (Silver Spring, Md.)

9462:

McNaughton, Robert. Unate truth functions. IRE Trans. **EC-10** (1961), 1–6.

This paper contains some applications of an elementary study of unate truth functions. One application is a method of deciding when a truth function is linearly separated, i.e., is expressible as a linear polynomial inequality in its arguments (letting 1 represent truth and 0 represent falsity). Other applications are to contact nets and to rectifier nets. Much of the material of this paper, although not in print, is well known to some logicians and switching theorists. The introductory sections (at least) should be interesting to anyone familiar with its use in computer circuitry. {There is a minor misprint in the last line on page 3: $F(\cdots, p_i, \cdots)$ should be $F(\cdots, \overline{p}_i, \cdots)$.} There is a good bibliography appended to the paper.

B. A. Galler (Ann Arbor, Mich.)

GENERAL MATHEMATICAL SYSTEMS

9463:

Marica, John G.; Bryant, Steven J. Unary algebras. Pacific J. Math. **10** (1960), 1347–1359.

Let A be a non-empty set and $f(a) = a'(a^0 = a, a^n = (a^{n-1})')$ a single unary operation on A. Then A is called a unary algebra. A^2 and B^2 denote the unary algebras formed from the Cartesian product of the sets A and B with themselves by defining the operation componentwise. The authors prove the following main theorem: If A and B are two finite unary algebras and A^2 is isomorphic to B^2, then $A \sim B$. There are proved a few other theorems of this type, e.g.: If A, B, C are connected finite unary algebras and $A \times C \sim B \times C$, then $A \sim B$. (If $x, y \in A$, then x is connected to y if and only if $x^n = y^m$ for some n and m. The relation, being an equivalence relation, divides A into equivalence classes called the components of A. If an algebra has only one component it is called connected.) In the infinite case the main theorem, in general, does not hold. However, the authors conjecture that the results obtained in the finite case can be generalized to some classes of infinite unary algebras.

J. W. Andrushkiw (Newark, N.J.)

9464:

Słomiński, J. On the embedding of abstract quasi algebras into equationally definable classes of abstract algebras. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **8** (1960), 11–17. (Russian summary, unbound insert)

A 'quasi-algebra' is an abstract algebra in the usual sense except that operations are defined only for arguments from specified subdomains. A homomorphism from A to B may then be 'almost onto': the image generates B, without necessarily being all of B. If L is a class of (full) algebras, an 'L free homomorphic image' of a quasi algebra A is an algebra B with a homomorphism from A into B that is a factor of any homomorphism from A into a B_1 in L. If L is an equational class, B is constructible from A in a natural way. Tensor products, over L, of algebras A and B are similarly characterized by reference to bilinear (that is, bihomomorphic) maps into algebras B_1 in L, and, again for equational classes, the tensor

product is constructed. These ideas are applied to the following situation. R is an algebra of the class L, with an additional associative (composition) operation (with neutral element e) that distributes over the other operations. An 'L module over R' is then an L algebra Z with additional operations a_r, r in R, providing a representation of R by endomorphisms of Z (as L algebra), with $a_e z = z$. If the a_r are defined only on subalgebras Z_r, then Z is a 'differential L algebra'. General theorems are obtained on the embedding of differential algebras in modules which include certain results of W. Słowikowski [same Bull. 7 (1959), 119–124; MR 21 #7230].

R. C. Lyndon (London)

THEORY OF NUMBERS

See also 9488, 9535, 9540.

9465:

Cohen, Eckford. The elementary arithmetical functions. Scripta Math. **25** (1960), 221–227.

It is shown how classical arithmetical identities such as Möbius' inversion formula, Ramanujan's formula and Hölder's formula are obtained from the author's general theory of even functions with respect to a modulus. [See #9540.]

H. Bergström (Göteborg)

9466:

Cohen, Eckford. The average order of certain types of arithmetical functions: generalized k-free numbers and totient points. Monatsh. Math. **64** (1960), 251–262.

Let M denote any set of positive integers and Z the set of all positive integers, $\rho_M(n)$ the characteristic function of the set M and $\mu_M(n)$ the inversion function of M given implicitly by the relation $\rho_M(n) = \sum_{d/n} \mu_M(d)$. Introducing $F_M(n) = \sum_{dc=n,\ c\in M} f(d)$ and $F(n) = F_Z(n)$ for any given arithmetical function $f(n)$, the author proves the identity $F_M(n) = \sum_{d/n} \mu_M(d) F(n/d)$, which is a generalization of Mertens identity. Using this identity and a related identity he gives new proofs for known estimations of the arithmetical functions considered in the Dirichlet divisor problems, Mertens totient problem and Gauss' circle problem. He also gives estimations of generalizations of these functions.

H. Bergström (Göteborg)

9467:

Scott, S. J. On the number of zeros of a cubic recurrence. Amer. Math. Monthly **67** (1960), 169–170.

Let $(T) = (T_0, T_1, T_2, \cdots)$ denote the cubic recurrence defined by the real numbers T_0, T_1, T_2 and the relation

$$T_{n+3} = PT_{n+2} - QT_{n+1} + RT_n \quad (n = 0, 1, \cdots),$$

where P, Q, R are real. When $T_k = 0$, k is said to be a zero of (T). It is proved that a recurrence (T), distinct from (T'): $T_n' = Uu^n(1-(-1)^n)$, can have no more than three zeros when the zeros of $x^3 - Px^2 + Qx - R$ are non-vanishing real numbers. This extends a result of Smiley [Amer. Math. Monthly **63** (1956), 171–172; MR **17**, 826].

L. Carlitz (Durham, N.C.)

9468:

Rotkiewicz, A. Sur le problème de Catalan. Elem. Math. **15** (1960), 121–124.

Theorem: Let x, y, z, t be integers greater than 1 such that (*) $x^z - y^t = 1$ but $(x, y, z, t) \neq (3, 2, 2, 3)$. Then each of x, y is divisible by two prime factors, one of which is greater than 11. Further, at least one prime divisor of x is congruent to 1 modulo t and at least one prime divisor of y is congruent to 1 modulo z. If $z = 2$, then y is also divisible by a prime congruent to 1 modulo t. The author concludes by giving a simple proof of the theorem of Moret-Blanc that the only solution of (*) with $x = t$ and $y = z$ is $(x, y, z, t) = (3, 2, 2, 3)$.

J. W. S. Cassels (Cambridge, England)

9469:

Lang, Serge. Some theorems and conjectures in diophantine equations. Bull. Amer. Math. Soc. **66** (1960), 240–249.

In this address to the American Mathematical Society in February 1960 the author discusses known results and unsolved problems, in particular the connection between local and global theories. The following subjects are discussed. (1) Hasse's theorem on quadratic forms. (2) Peck's theorem on forms of arbitrary degree in a large number of variables over a totally imaginary ground field and forms over quasi-algebraically closed fields. (3) Principal homogeneous spaces, and the connection to cohomology theory. (4) The special case of curves, in particular the theorems of Siegel and of Mordell-Weil. A (rather incomplete) list of 27 papers closes the report.

K. Mahler (Manchester)

9470:

Mordell, L. J. On a pellian equation conjecture. Acta Arith. **6** (1960), 137–144.

Let p be a prime $\equiv 1 \pmod 4$ and let $\varepsilon = \frac{1}{2}(t + u\sqrt{p}) > 1$ be the fundamental unit in the quadratic field $R(\sqrt{p})$. It has been conjectured by Ankeny, Artin and Chowla [Ann. of Math. (2) **56** (1952), 479–493; MR **14**, 251] that $u \not\equiv 0 \pmod p$.

The following results are proved in the present paper. (1) If p is a regular prime (that is, the number of classes of ideals in the cyclotomic field $R(e^{2\pi i/p})$ is not divisible by p), then $u \not\equiv 0 \pmod p$. (2) If $p \equiv 5 \pmod 8$, then $u \equiv 0 \pmod p$ if and only if the Bernoulli number $B_{(p-1)/4} \equiv 0 \pmod p$.

L. Carlitz (Durham, N.C.)

9471:

Mordell, L. J. Recent work in number theory. Scripta Math. **25** (1960), 93–103.

The author discusses some recent results in number theory, most of which is his own work. (1) A simple proof that the multinomial coefficients are integral. (2) Let (u, t) denote the fundamental solution of the equation $y^2 - Dx^2 = -4$. The author has proved [see #9470] that if p is a prime $\equiv 1 \pmod 4$ and the number of classes of ideals in the cyclotomic field $k(e^{2\pi i/p})$ is not divisible by p, then $u \not\equiv 0 \pmod p$; whereas if p is a prime $\equiv 5 \pmod 8$, then $u \equiv 0 \pmod p$ if and only if the Bernoulli number $B_{(p-1)/4} \equiv 0 \pmod p$. Chowla has proved that the latter result holds for all prime $p \equiv 1 \pmod 4$. (3) Discussion of the following problem of J. J. Schoenberg. Given any quadrilateral $PQRS$, do there exist rational quadrilaterals

$P'Q'R'S'$ (that is, quadrilaterals with rational sides and diagonals) whose vertices P', Q', R', S' are arbitrarily near to P, Q, R, S? (4) Integral solutions of two simultaneous homogeneous quadratic equations with rational coefficients. The author has proved the following result [Math. Sem. Univ. Hamburg Abh. **23** (1959), 126–143; MR **21** #3378]. Let $f(x)$, $g(x)$ be two quadratic forms with rational coefficients in n essential variables. Suppose that for real λ, μ not both zero, no member of the pencil $\lambda f(x)+\mu g(x)$ is either definite of rank $\leq n$ or singular of rank <5 but that at least one member has more than four positive and more than four negative squares when expressed as a sum of squares of real linear forms. Then the system $f(x)=0$, $g(x)=0$ has an infinity of integral solutions with non-zero values of the variables when $n\geq 13$. (5) Let $L_1, \cdots, L_n$ be linear forms in $u_1, \cdots, u_n$ with real coefficients and determinant 1 and let $C_1, \cdots, C_n$ be any real numbers. Put $M=\inf \prod_{r=1}^{n} |L_r+C_r|$, that is, the lower bound for all integers (u). The author has proved [J. London Math. Soc. **35** (1960), 91–97; MR **22** #1564] that

$$M \leq 2^{-n/2}\{4-2(2-\tfrac{3}{4}\sqrt{2})^n-2^{-n/2}\}^{-1}.$$

L. Carlitz (Durham, N.C.)

9472:

Makowski, A.; Schinzel, A. Sur l'équation indéterminée de R. Goormaghtigh. Mathesis **68** (1959), 128–142.

R. Goormaghtigh conjectured that the only solution of $N=1+x+\cdots+x^m=1+y+\cdots+y^n$, $m>n>1$, $y>x>1$, are $N=31$, $y=5$, $x=2$, $m=4$, $n=2$ and $N=8191$, $y=90$, $x=2$, $m=12$, $n=2$. The authors prove several special cases of this conjecture. *P. Erdös* (Budapest)

9473:

Lewandowski, Z. Proof of a special case of Fermat's last theorem. Wiadom. Mat. (2) **2**, 249–252 (1959). (Polish)

The author proves the following theorem: if x, y, z, n are integers, $n>2$ and $x\leq 2n$ or $y\leq 2n$ then $x^n+y^n\neq z^n$. A stronger theorem has been proved, however, in a more complicated way, by K. Inkeri [Ann. Univ. Turku. Ser. A **16** (1953), no. 1; MR **15**, 401]. *S. Knapowski* (Poznań)

9474:

Rotkiewicz, A. Une remarque sur le dernier théorème de Fermat. Mathesis **69** (1960), 135–140.

By an elementary method the author proves that if n is an integer >2, and the positive integers x, y, z satisfy $x^n+y^n=z^n$, then $x>3^n$ and $y>3^n$. The author distinguishes four cases: (I) n is odd, and (1) at least one of $(x+y,\ n)$, $(z-x,\ n)$, $(z-y,\ n)$ is greater than 1, or (2) $(x+y,\ n)=(z-x,\ n)=(z-y,\ n)=1$; (II) $n=2n_1$, where n_1 is odd, since Fermat's last theorem is true for $n=4n_2$, and (1) at least one of $(x^2+y^2,\ n/2)$, $(z^2-x^2,\ n/2)$, $(z^2-y^2,\ n/2)$ is greater than 1, or (2) $(x^2+y^2,\ n/2)=(z^2-x^2,\ n/2)=(z^2-y^2,\ n/2)=1$; and in each case applies the following lemma: Let a and b be relatively prime integers, and let p be an odd prime; also let n be a positive odd integer. Let α be an integer ≥ 0, and let λ be an integer ≥ 1. Then $p^\lambda \mid a\pm b$, $p^{\lambda+1}\nmid a\pm b$, $p^\alpha \mid n$, $p^{\alpha+1}\nmid n$ imply $p^{\lambda+\alpha}\mid a^n\pm b^n$, $p^{\lambda+\alpha+1}\nmid a^n\pm b^n$. *T. Morishima* (Tokyo)

9475:

Watson, G. L. ★Integral quadratic forms. Cambridge Tracts in Mathematics and Mathematical Physics, No. 51. Cambridge University Press, New York, 1960. xii+143 pp. $5.00.

This monograph is chiefly concerned with equivalence and representation of integers by quadratic forms with integral coefficients. A background of only elementary number theory and the rudiments of matrix theory is assumed. The methods are arithmetical and at times most ingenious. The first five chapters are mostly classical (if rational equivalence comes under that heading) and have much in common with the reviewer's *The arithmetic theory of quadratic forms* [Math. Assoc. of America, Buffalo, N.Y., 1950; MR **12**, 244]. They are concerned with reduction, the rational invariants (Hilbert and Hasse symbols), p-adic equivalence from the point of view of congruences, the congruence class, and genus.

The reader will notice one change in the beginning, wherein the quadratic form is written: $\frac{1}{2}x'Ax$, where the matrix A has elements $2a_{ii}$ along the diagonal and a_{ij} elsewhere. This is to prepare for the following definition of the discriminant of the form:

$$d = d(f) = (-1)^{n/2}|A| \qquad (n \text{ even}),$$
$$= \tfrac{1}{2}(-1)^{(n-1)/2}|A| \quad (n \text{ odd}).$$

This definition simplifies other notations later.

The fifth chapter contains the author's beautiful result that an indefinite form in four or more variables represents properly an integer a if it represents it properly modulo d.

Chapter 6, which deals with rational transformations and automorphs, is largely preparation for the following chapter.

In chapter 7, the spinor norm of Eichler is defined in arithmetical terms. To accomplish this the author defines the weight, $w(R)$, of a rational transformation R, to be the least positive integer such that $w(R)e$ is integral when e ranges over all the minor determinants of R. The norm of R is the unique square-free positive integer such that the product of the norm and $w(R)$ is a square. Then two forms in n variables are said to be in the same spinor genus if one may be taken into the other by a rational transformation R whose determinant is $\neq 1$, whose norm is 1, and such that all the denominators of the elements of R are prime to d, the discriminant of the form. He then proceeds to prove the fundamental theorem: Two indefinite forms in three or more variables are equivalent if and only if they are in the same spinor genus. Simpler congruence conditions are given for spinor-relatedness.

In chapter 7 it is also shown that if f is a positive form in four or more variables, with discriminant d, and a is a positive integer such that the congruence $f(x)\equiv a \pmod d$ has a solution with x integral and primitive, then some form f' in the spinor genus of f represents a properly. For positive ternary quadratic forms there are so-called "exceptional" integers a which do not have this property. It is shown that if a is exceptional, ap^2 is exceptional for every prime p not dividing the discriminant.

Chapter 8 is concerned with factorization of the general rational automorph into reflexions with denominators prime to a given odd integer m. In particular, every rational automorph of a form with discriminant d, whose weight is prime to d, has the same norm as some product of rational reflexions of f, each reflexion having denominator prime to d.

Though in a number of spots this tract is very difficult reading, it is a notable contribution to the literature of quadratic forms and brings one to the threshold of much that is still unknown in the subject.

B. W. Jones (Boulder, Colo.)

9476:

Erdös, P. Über die kleinste quadratfreie Zahl einer arithmetischen Reihe. Monatsh. Math. **64** (1960), 314–316.

Let a, d be coprime integers such that $1 \leqq a < d$, and denote by $q(a, d)$ the smallest positive square-free number which is $\equiv a \pmod{d}$. It was shown by K. Prachar [Monatsh. Math. **62** (1958), 173–176; MR **19**, 1160] that, for suitable absolute constants c_1, c_2,

$$q(a, d) \leqq c_1 d^{3/2} 2^{c_2 \log d/\log\log d}.$$

In the present paper this estimate is improved to

$$q(a,.d) \leqq c_3 d^{3/2}/\log d,$$

where c_3 is again an absolute constant. The author conjectures that, in fact, $q(a, d) = o(d^{1+\varepsilon})$ and states that an easy argument leads to the conclusion

$$q(a, d) = \Omega(d \log d/\log\log d).$$

L. Mirsky (Sheffield)

9477:

Chandrasekharan, Komaravolu; Narasimhan, Raghavan. Sur l'ordre moyen de quelques fonctions arithmétiques. C. R. Acad. Sci. Paris **251** (1960), 1333–1335.

Defining $\Delta(x)$ by $\sum_{n=1}^{x} d(n) = x \log x + (2\gamma - 1)x + \Delta(x)$, where $d(n)$ is the number of divisors of n and γ is Euler's constant, the authors refer to the Ω-theorem that $\Delta(x)/x^{1/4}$ has upper and lower limits $+\infty$ and $-\infty$ as $x \to \infty$. They note that the generating function $\varphi(s) = \zeta^2(s) = \sum d(n) n^{-s}$ of the coefficients $d(n)$ satisfies a functional equation (1) $\Gamma^2(\frac{1}{2}s)\varphi(s) = \Gamma^2(\frac{1}{2} - \frac{1}{2}s)\psi(1-s)$, where $\varphi(s)$ and $\psi(s)$ have Dirichlet expansions $\varphi(s) = \sum a_n \lambda_n^{-s}$ and $\psi(s) = \sum b_n \mu_n^{-s}$ in certain half-planes. They then observe that there are other solutions of (1) (e.g., ζ-functions of real quadratic fields), and enunciate Ω-theorems for the error terms in asymptotic formulae for the Riesz sums $\sum_{\lambda_n \leqq x} a_n (x - \lambda_n)^\rho$ in a wide class of such cases. {The second sentence in the statement of Theorem I is somewhat obscure. The intended meaning seems to be: Suppose that the sequence $\{\lambda_n\}$ contains a part $\{\lambda_{nk}\}$ such that no $\lambda_n^{1/2}$ can be represented as a linear combination of the numbers $\{\lambda_{nk}^{1/2}\}$ with coefficients ± 1, unless $\lambda_n^{1/2} = \pm \lambda_{nr}^{1/2}$ for some r, in which case $\lambda_n^{1/2}$ has no other representation. It seems to the reviewer, however, that this condition should relate to the sequence $\{\mu_n\}$ rather than $\{\lambda_n\}$.}

A. E. Ingham (Cambridge, England)

9478:

Rényi, A. Probabilistic methods in number theory. Proc. Internat. Congress Math. 1958, pp. 529–539. Cambridge Univ. Press, New York, 1960.

This is an exposition of recent work in probabilistic number theory. Topics mentioned include the proof by P. Turán and the author [Acta Arith. **4** (1958), 71–84; MR **20** #3112] of a conjecture of the reviewer, Linnik's large sieve, statistical properties of digits in various expansions of real numbers, and the additive theory of random sequences of integers.

W. J. LeVeque (Ann Arbor, Mich.)

9479:

Ankeny, N. C. Criterion for rth power residuacity. Pacific J. Math. **10** (1960), 1115–1124.

Let r be an odd prime. Let p and q be primes such that $p \equiv q \equiv 1 \pmod{r}$. Let $\sum_i^{(k)}$ denote the sum over all $j_1, \cdots, j_{k+1} = 1, 2, \cdots, r-1$, such that $j_1 + \cdots + j_k - kj_{k+1} \equiv i \pmod{r}$. Consider the following system of Diophantine equations in $r-1$ unknowns:

$$(1) \qquad r \sum_{j=1}^{r-1} X_j^2 - \Big(\sum_{j=1}^{r-1} X_j\Big)^2 = (r-1)p^{r-2},$$

$$(2) \qquad \sum_1^{(1)} X_{j_1} X_{j_2} = \sum_i^{(1)} X_{j_1} X_{j_2}, \quad 2 \leqq i \leqq \tfrac{1}{2}(r-1),$$

$$(3) \qquad 1 + \sum_{j=1}^{r-1} X_j \equiv \sum_{j=1}^{r-1} j X_j \equiv 0 \pmod{r},$$

(4) not all of the $X_j \equiv 0 \pmod{p}$ and

$$\sum_i^{(k)} X_{j_1} \cdots X_{j_{k+1}} \equiv \sum_0^{(k)} X_{j_1} \cdots X_{j_{k+1}} \pmod{p^{r-k-1}}$$

for $2 \leqq k \leqq r-2$, $1 \leqq i \leqq r-1$. This system is shown to have exactly $r-1$ distinct solutions in rational integers. Let $\{X_j = a_j,\ 1 \leqq j \leqq r-1\}$ be any one of these solutions, and let h be any integer such that $h^r \equiv 1 \pmod{q}$, $h \not\equiv 1 \pmod{q}$. Then q is an rth power modulo p if and only if $p \sum a_j h^j$ is an rth power modulo q. Moreover, r is an rth power (mod p) if and only if $\sum j a_j + \frac{1}{2} r a_{r-1} \equiv 0 \pmod{r^2}$. The determination of solutions of the system (1) through (4) is formidable, even for moderate values of r. {There are two misprints in the statement of the main result, Theorem B.}

W. H. Mills (Berkeley, Calif.)

9480:

Whiteman, A. L. The cyclotomic numbers of order twelve. Acta Arith. **6** (1960), 53–76.

The author considers cyclotomic numbers of order e, denoted by $(i, j)_e$, meaning the number of solutions in y of $y \equiv g^{et+i}$, $1 + y \equiv g^{et+j} \pmod{p}$, for $p - 1 = ef$, and g a fixed primitive root of p. Exact formulas have been given for $e = 2, 3, 4, 5, 6, 8$, and 16; the present paper settles the case $e = 12$, completing work of Dickson [Amer. J. Math. **57** (1935), 391–424]. The values depend on the components in the decompositions: $p = x^2 + 4y^2 = A^2 + 3B^2$. The nonexistence of residue difference sets is established as in the author's earlier work [Trans. Amer. Math. Soc. **86** (1957), 401–413; MR **19**, 1160].

H. Cohn (Tucson, Ariz.)

9481:

Staś, W. Über einige Abschätzungen in Idealklassen. Acta Arith. **6** (1960), 1–10.

The author here extends the methods of his paper in Acta Arith. **5** (1959), 179–195 [MR **21** #6355] to cover the case of the distribution of prime ideals in ideal classes.

Let $\mathfrak{f}$ be a fixed ideal of an algebraic number field F, $\mathscr{H}$ an ideal class mod $\mathfrak{f}$, $\mathscr{H}_0$ the principal class mod $\mathfrak{f}$, $h(\mathfrak{f})$ the class number mod $\mathfrak{f}$, χ a character mod $\mathfrak{f}$ and $\zeta(s, \chi)$ Landau's zeta function belonging to the character χ.

Let

$$\Delta(x, \mathscr{H}) = \sum_{\substack{N(\mathfrak{y})^m \leqq x \\ \mathfrak{y}^m \in \mathscr{H}}} \log N(\mathfrak{y}) - \frac{x}{h(\mathfrak{f})};$$

let A_χ be the set of non-trivial zeros of all $\zeta(s, \chi)$, $\rho=\beta+i\gamma$ a member of A_χ, and $\varepsilon(\tau)=\mathrm{Max}_{|\gamma|\leq\tau}\beta$. Then, using classical methods, the author proves that for T suitably chosen, we have the following upper bound:

$$\mathop{\mathrm{Max}}_{1\leq x\leq T} |\Delta(x, \mathscr{H})| \leq T^{\varepsilon(T+1)} \log^4 T.$$

Using the method of Turán [*Eine neue Method in der Analysis und deren Anwendungen*, Akadémiai Kiadó, Budapest, 1953; MR **15**, 688] the author derives the following lower bound:

$$\mathop{\mathrm{Max}}_{1\leq x\leq T} |\Delta(x, \mathscr{H}_0)| >$$

$$T^{\varepsilon \exp\sqrt{\log\log\log T}} \exp\left(-7\,\frac{\log T \log\log\log T}{\log\log T}\right)$$

and then infers that for suitable T

$$\mathop{\mathrm{Max}}_{1\leq x\leq T} |\Delta(x, \mathscr{H})| \leq$$

$$T^{\delta(T)} \exp\left(8\,\frac{\log T \log\log\log T}{\log\log T}\right) \mathop{\mathrm{Max}}_{1\leq x\leq T} |\Delta(x, \mathscr{H}_0)|$$

where $\delta(T)=\varepsilon(T+1)-\varepsilon \exp\sqrt{\log\log\log T}$.

R. Ayoub (University Park, Pa.)

9482:

Schmidt, Wolfgang. A metrical theorem in diophantine approximation. Canad. J. Math. **12** (1960), 619–631.

The author proves the following more precise version of an old theorem of Khintchine (Hinčin) [Math. Z. **24** (1926), 706–714]:

Let $\psi_1(q), \cdots, \psi_n(q)$ be n non-negative functions of the positive integer q and suppose that

$$\psi(q) = \prod_{1\leq i\leq n} \psi_i(q) \tag{1}$$

is monotone decreasing. For real numbers $\theta_1, \cdots, \theta_n$ denote by $N(h; \theta_1, \cdots, \theta_n)$ the number of integral solutions of

$$0 \leq q\theta_i - p_i < \psi_i(q) \quad (1 \leq i \leq n) \tag{2}$$

with $1 \leq q \leq h$, and put $\Psi(h)=\sum_{1\leq q\leq h}\psi(q)$, $\Omega(h)=\sum_{1\leq q\leq h} q^{-1}\psi(q)$. Then

$$N(h; \theta_1, \cdots, \theta_n) = \Psi(h) + O(\Psi^{1/2}(h)\Omega^{1/2}(h)\log^{2+\varepsilon}\Psi(h)) \tag{3}$$

for almost all sets $\theta_1, \cdots, \theta_n$.

Khintchine's theorem is, substantially, the statement that $N(h; \theta_1, \cdots, \theta_n)$ is bounded for almost all $\theta_1, \cdots, \theta_n$ as $h\to\infty$ if and only if $\Psi(h)$ is bounded.

The author also indicates the proof of a similar theorem in which $q\theta_i$ and $\psi_i(q)$ in (2) are replaced respectively by a linear form $\theta_{i1}q_1+\cdots+\theta_{im}q_m$ in integers $q_1, \cdots, q_m$, and by a non-negative function $\psi_i(q_1, \cdots, q_m)$ respectively. In this second theorem there is no analogue of the condition that $\psi(q)$ be monotone but, presumably, the explicit error term that is given in the analogue of (3) is worse than that in (3).

The proofs require considerable modification of pre-existing techniques.

J. W. S. Cassels (Cambridge, England)

9483:

Newman, Morris. Irrational power series. Proc. Amer. Math. Soc. **11** (1960), 699–702.

The author proves the following theorem. Let α be a real number. Let g denote a polynomial of degree ≥ 1. Then the power series $\sum_{n=0}^{\infty} g([n\alpha])x^n$ defines a rational function of x if and only if α is a rational number.

The proof uses the fact that if α is irrational the fractional parts of $n\alpha$ $(n=1, 2, 3, \cdots)$ are everywhere dense in the unit interval.

As a special case it easily follows that, when $\alpha>0$, the power series $\sum_{n=0}^{\infty} x^{[n\alpha]}$ represents a rational function if and only if α is rational. *J. Popken* (Amstelveen)

9484:

Macbeath, A. M. Abstract theory of packings and coverings. I. Proc. Glasgow Math. Assoc. **4**, 92–95 (1959).

The author considers the measure space (X, S, μ) and a countable group G of permutations of X, each of which is measure-preserving. e denotes an identical permutation. A set $P\in S$ is a G-packing if $P\cap gP=\varphi$ for all $g\neq e$, $g\in G$; and a G-covering if $\bigcup\{gP: g\in G\}=X$; P is a fundamental domain if it is both a G-covering and a G-packing. The author shows that the measure of any G-covering is greater than or equal to the measure of any G-packing and uses this result to show that if X is a metric space such that the measure of a sphere of radius $r>\inf\mu(C)$ for C any G-covering then there exists $g\in G$, $g\neq e$ such that the distance between the centre of the sphere and its image under g is less than $2r$. The author also proves a somewhat similar result when X is a group and G is a sub-group acting on X by left translation. A number of applications are given which include results of Minkowski, Siegel, Blichfeldt and Santaló.

H. G. Eggleston (London)

9485:

Świerczkowski, S. Abstract theory of packings and coverings. II. Proc. Glasgow Math. Assoc. **4**, 96–100 (1959).

The author establishes necessary and sufficient conditions for the existence of a fundamental domain, using the terminology of the previous review. The measure space (X, S, μ) has a fundamental domain if and only if the following three properties hold: (1) if $A\in S$, $g\in G$, $\mu(A)>0$, and $g\neq e$, then there exists $B\subset A$ such that $\mu(B)>0$ and $B\cap gB=\varphi$; (2) no $g\neq e$ has fixed points; (3) if $A\in S$ has arbitrarily small coverings then $\mu(A)=0$.

The author also gives further results of a similar nature and an example to show that his criterion is valid only when the measure μ is σ-finite and complete.

H. G. Eggleston (London)

FIELDS

See also 9495.

9486:

Mathews, G. B. ★Algebraic equations. Revised by W. E. H. Berwick. Reprinting of Cambridge Tracts in Mathematics and Mathematical Physics, No. 6. Hafner Publishing Co., New York, 1960. vii+69 pp. $3.00.

A reprint of the 1930 edition [3rd ed., Cambridge Univ. Press, London].

9487:

Borofsky, S. Convergence in ordered integral domains. Amer. Math. Monthly **67** (1960), 221–227.

The author studies the possibilities of permutability of the limit of a sequence with the product or quotient of sequences in an ordered non-archimedean valued ring. He obtains also that the valuation of a ring is archimedean if and only if $0<a<1$ does imply $a^n\to 0$, and gives some conditions of limitability of sequences by Cesàro means. {It must be pointed out that the definition of convergence given by the author on page 221 is actually the definition of limit of a sequence and that these definitions should not, generally, be equivalent.} *P. Abellanas* (Madrid)

9488:

Yamamoto, Koichi. The Artin-Hasse-Šafarevič function. Japan. J. Math. **29** (1959), 165–172.

An account, with some new contributions, of the main results on the Artin-Hasse (et al.) series $E(X)=\sum_{i=0}^{\infty} p^{-i}X^{p^i}$ (p a fixed prime). The results are stated for fields of characteristic zero, and those which involve power series are stated for the completion Ω of an algebraically closed field with respect to a valuation of rank 1, with residue field of characteristic p. Since the usefulness of $E(X)$ stems from the possibility of reducing mod p, attention is paid to whether the coefficients of the various power series are integers of Ω. In this context, Proposition 7 gives a useful criterion of deciding whether, given an element $f=\sum_{n=1}^{\infty} n^{-1}a_nX^n$ of $X\Omega\{X\}$, the power series $\exp f$ has coefficients which are integers of Ω. The main result, however, from which many of the basic algorithms on Witt vectors follow, is probably the following: Let k be a field of characteristic zero, let $\{z_0, z_1, \cdots\}$ be a sequence of elements of k, and let $\theta: x\to x^p$ be a field-isomorphism of k into k; let a be a Witt vector with ghost components $a^{(n)}=z_0{}^{p^n}+pz_1{}^{p^{n-1}}+\cdots+p^nz_n$; let Δ_s ($s=1, 2, \cdots$) be the mapping $x\to\Delta_s x=p^{-s}(x^{p^s}-x^{p^{s-1}})$ of k into k; then the components of a are given by $a_0=z_0$, $a_n=z_n+\Delta_1a_{n-1}+\cdots+\Delta_na_0$. {It is also stated that the a_n are "polynomials in the z's, with rational coefficients"; but this will be true only if θ satisfies additional conditions, as one readily sees by taking the complex field for k, and the complex conjugation for θ.}

I. Barsotti (Providence, R.I.)

9489:

Goldman, Lawrence. Solutions of first order differential equations which are solutions of linear differential equations of higher order. Proc. Amer. Math. Soc. **10** (1959), 936–939.

Let F be an ordinary differential field of characteristic zero. The author considers a solution z of an algebraic differential equation over F of order one which is also a solution of a linear differential equation over F. He proves, among other results, that there exist an algebraic extension F_0 of F and an element $u\in F_0(z, z')$, such that z is algebraic over $F(u)$ and that u satisfies a linear differential equation over F_0 of order one.

E. R. Kolchin (New York)

ABSTRACT ALGEBRAIC GEOMETRY

See also 9488, 9515.

9490:

Abhyankar, Shreeram. Cubic surfaces with a double line. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **32** (1960), 455–511.

The main purpose of this paper is to establish various preliminary results that the author required in two recent papers on the fundamental groups of plane algebraic curves [Amer. J. Math. **82** (1960), 365–373, 374–388; MR **22** #1577, 1578].

If the ground-field is algebraically closed and has characteristic $p\neq 2, 3$, there are in P^3 exactly four projectively distinct types of cubic surface with a double line. Of these surfaces two are cones (one with a cuspidal line), and the others are cubic scrolls which are referred to here as special (F_3) and non-special (F_4) cubic surfaces with a double line.

If F_3 or F_4 is projected onto a plane from a point in general position the branch locus is a three-cuspidal quartic. If the vertex of projection is special, then the branch locus is a cuspidal cubic together with its flex tangent.

A birational representation of F_4 as a projective plane is obtained by projecting it from a pinch point of its double line, and via a special quadratic transformation of the plane a normalization of F_4 is represented as an immediate quadratic transform of a projective plane. F_3 is similarly treated by projecting it from the pinch point of its double line onto a plane. From these representations it follows that the normalizations of F_3 and F_4 are biregularly equivalent. *D. Kirby* (Leeds)

9491:

Murre, J. P. On a uniqueness theorem for certain kinds of birational transformations. Nederl. Akad. Wetensch. Proc. Ser. A **62** = Indag. Math. **21** (1959), 129–134.

Let $T: V\to V'$ be a birational transformation between two n-dimensional complete varieties; assume T to be antiregular at W, where W is a component of the fundamental locus of T on V. Suppose $T\{W\}$ to have only one $(n-1)$-dimensional component W', and every point of W(resp. W') to be simple on V(resp. V'). Denoted by $|\overline{W}|$ the point $|W|-\bigcup_\lambda|\overline{W}_\lambda|$. The author shows that T and the monoidal dilatation of V with centre W are biregularly equivalent at all points of $|\overline{W}|$. The main result states that any antiregular birational transformation is a finite product of locally quadratic transformations.

K.-Y. Hsü (Taipei)

9492:

Roquette, Peter. Some fundamental theorems on abelian function fields. Proc. Internat. Congress Math. 1958, pp. 322–329. Cambridge Univ. Press, New York, 1960.

Topics are picked up from the author's papers in Jber. Deutsch. Math. Verein. **60** (1957) Abt 1, 1–21 [MR **21** #3405], and J. Reine Angew. Math. **197** (1957), 49–67 [MR **19**, 981]. These topics are related with the celebrated finiteness theorem of Severi-Néron. *M. Nagata* (Kyoto)

9493:

Serre, Jean-Pierre. Groupes proalgébriques. Inst. Hautes Études Sci. Publ. Math. No. 7 (1960), 67 pp.

Cet article est consacré à l'étude des systèmes projectifs de groupes algébriques. Pour justifier cette étude, l'auteur cite trois systèmes projectifs dont l'étude s'est récemment imposée: les groupes isogènes au-dessus d'un groupe algébrique fixé, les jacobiennes généralisées d'une courbe fixée; enfin, soit U le groupe des unités d'un corps K extension finie du corps des nombres p-adiques; pour tout $n \geq 1$, les unités x telles que $x-1$ soit de valuation $\geq n$ forment un sous-groupe U_n de U, et le quotient U/U_n est l'ensemble des points rationnels d'une variété de groupe G_n définie sur le corps premier $\mathbf{F}_p$; dans un article annoncé comme la suite de celui-ci, l'auteur a appliqué ce dernier système projectif à l'étude du corps de classes locals.

Nous commencerons par énoncer les principaux résultats sans utiliser le langage des systèmes projectifs, mais en se plaçant dans le cadre de l'algèbre des catégories; on se donne un corps k algébriquement clos de caractéristique p, et les variétés sont des variétés au sens de l'article connu de l'auteur sur les faisceaux algébriques cohérents [Ann. of Math. (2) **61** (1955), 197–278; MR **16**, 953]. Soient G et H deux variétés de groupe commutatives (hypothèse non répétée); un morphisme de G dans H est une application régulière f telle que $f(x)\cdot f(y)=f(xy)$; si $p\neq 0$, il faut introduire les quasi-morphismes si l'on veut éviter les difficultés dues à l'inséparabilité: un quasi-morphisme est une application f de G dans H telle que $f(x)\cdot f(y)=f(xy)$ et que $(\varphi\circ f)^q$ soit régulière sur l'ouvert $f^{-1}(U)$ de G pour tout ouvert U de H, toute fonction numérique régulière φ sur U, et toute puissance q de p assez grande; enfin, lorsque $p=0$, les quasi-morphismes seront simplement les morphismes. Enfin, on notera $\mathscr{H}(G, H)$ l'ensemble des symboles $f\cdot\alpha^{-1}$ avec $f: A\to H$ et une isogénie $\alpha: A\to G$; l'égalité $f\cdot\alpha^{-1}=f'\cdot\alpha'^{-1}$ signifie qu'il existe des isogénies $\beta: B\to A$ et $\beta': B\to A'$ telles que $f\cdot\beta=f'\cdot\beta'$ et $\alpha\beta=\alpha'\beta'$. L'introduction des groupes $\mathscr{H}(G, H)$ généralise une construction de Weil pour les variétés abéliennes, et permet de "négliger les isogénies".

Les groupes algébriques et les quasi-morphismes forment une catégorie abélienne $\mathscr{Q}$ ("exact category" au sens de Buchsbaum; cf. H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton University Press, Princeton, 1956 [MR **17**, 1040]). Par les méthodes de N. Yoneda [J. Fac. Sci. Univ. Tokyo. Sect. I. **7** (1954), 33–67; MR **15**, 731], on peut définir les groupes $\mathrm{Ext}^n(G, H)$ dans la catégorie $\mathscr{Q}$. La plus grande partie de l'article de l'auteur est consacrée à la détermination explicite de ces groupes; on introduit d'abord les groupes élémentaires: groupe fini $\mathbf{F}_l$ d'ordre premier l, groupe additif G_a, groupe multiplicatif G_m, variétés abéliennes simples; puis on prouve que tout groupe algébrique admet une suite de composition dont les quotients successifs sont des groupes élémentaires, enfin que l'on a $\mathrm{Ext}^n(G, H)=0$ si G et H sont élémentaires et $n\geq 2$, sauf lorsque $p\neq 0$, $n=2$, $G=G_a$ et $H=\mathbf{F}_p$.

On peut maintenant énoncer les résultats les plus importants de l'article. Tout d'abord (théorème 10.1), on a $\mathrm{Ext}^n(G, H)=0$ quels que soient G et H si $p=0$ et $n\geq 2$, ou $p\neq 0$ et $n\geq 3$. Puis, (théorème 10.2), pour tout quasi-morphisme $f: G\to H$ de noyau fini, il existe un groupe algébrique G_1 contenant G comme sous-groupe fermé, et un quasi-morphisme $f_1: G_1\to H$ surjectif, de noyau fini et prolongeant f. Enfin, les foncteurs covariants exacts de $\mathscr{Q}$ dans la catégorie des groupes abéliens "discrets" sont les produits des foncteurs suivants: (a) $T_l(G)$: ensemble des points de G dont l'ordre est une puissance du nombre premier l; (b) $\mathscr{H}(G_m, G)$; (c) $\varinjlim_n \mathscr{H}(W_n, G)$, où W_n est le groupe des vecteurs de Witt de longueur n (à remplacer par $\mathscr{H}(G_a, G)$ si $p=0$); (d) $\mathscr{H}(A, A(G))$, où A est une variété abélienne simple et $A(G)$ le quotient de G par son plus grand sous-groupe linéaire fermé connexe.

Donnons maintenant quelques indications sur les méthodes utilisées. On sait que la géométrie algébrique de l'auteur est fondée sur la notion d'espace annelé (espace topologique avec un faisceau convenable de fonctions continues). Pour tout groupe algébrique G, on appelle groupe quasi-algébrique associé le groupe G muni du faisceau des fonctions $f^{p^{-n}}$ si $p\neq 0$ (f si $p=0$), où f est une fonction régulière numérique sur un ouvert de G, et n un entier ≥ 0 arbitraire. Les quasi-morphismes $f: G\to H$ sont alors les morphismes pour les groupes quasi-algébriques associés. Un groupe proalgébrique est un groupe muni d'une structure d'espace annelé, qui peut se représenter comme limite projectives de groupes quasi-algébriques. La catégorie $\mathscr{P}$ des groupes proalgébriques est une "bonne" catégorie (avec limites projectives exactes) et on peut lui appliquer les méthodes de l'algèbre homologique (foncteurs dérivés). On définit $\pi_0(G)$ comme le quotient de G par son plus grand sous-groupe fermé connexe, et les π_i sont les foncteurs dérivés de π_0; on peut aussi définir les groupes simplement connexes, et le revêtement universel $\bar{G}$ d'un groupe proalgébrique G, en analogie avec la théorie des groupes topologiques. Le théorème 10.2 s'exprime alors en disant que $\pi_i(G)=0$ pour $i\geq 2$, ou que $\bar{G}$ est un foncteur exact en G (théorème 10.3). De plus, la catégorie $\mathscr{Q}$ est formée d'objets vérifiant la condition de chaînes descendante; cela permet d'appliquer à la catégorie $\mathscr{P}$ des limites projectives d'objets de $\mathscr{Q}$ les méthodes de P. Gabriel [Séminaire Dubreil-Jacotin-Pisot 1958–59, Paris, 1960; Exp. 17]. Par conséquent, pour tout foncteur exact T de $\mathscr{Q}$ dans la catégorie des groupes abéliens, il existe un groupe proalgébrique P défini à isomorphisme près tel que $T(G)\cong\mathrm{Hom}(P, G)$; l'étude des foncteurs exacts sur $\mathscr{Q}$ est donc ramenée à celle des objets projectifs dans $\mathscr{P}$, et ces derniers se décomposent de manière unique en produit (éventuellement infini) de projectifs indécomposables.

{Nous terminerons par quelques remarques. La catégorie $\mathscr{Q}$ des groupes quasi-algébriques est étudiée de manière complète dans le présent article; mais on sait que les groupes algébriques et les morphismes forment une catégorie $\mathscr{A}$ qui n'est pas abélienne. Selon l'auteur, A. Grothendieck aurait montré que les schémas de type fini en groupes commutatifs sur le corps k forment de nouveau une catégorie abélienne $\mathscr{A}'$; de plus, tout schéma de type fini en groupes contient un plus grand sous-grand sous-groupe algébrique, et le quotient est un "groupe infinitésimal" de l'espèce étudiée par J. Dieudonné dans de nombreux articles (voir aussi P. Gabriel [Séminaire J.-P. Serre, 1960, Collège de France], P. Cartier [Bull. Soc. Math. France **87** (1959), 191–220; MR **22** #6813]). Nous noterons H_n le noyau de la puissance n-ième de l'homomorphisme de Frobenius F dans le groupe G_m, et $W_{n,r}$ le noyau de F^n dans W_r. On peut définir deux nouveaux foncteurs sur $\mathscr{A}'$, à savoir $\varinjlim_n \mathrm{Hom}(H_n, G)$ et $\varinjlim_{n,r} \mathrm{Hom}(W_{n,r}, G)$; il conviendrait de montrer que ces foncteurs sont exacts, et qu'il n'y a pas d'autre foncteur exact "élémentaire" sur $\mathscr{A}'$ que ceux-ci et les foncteurs précédemment décrits (pour la catégorie $\mathscr{Q}$). Cette étude

ne semble pas encore avoir été entre prise; il ne devrait pas y avoir de difficulté sérieuse à étendre à $\mathscr{A}'$ les résultats démontrés par Serre pour la catégorie $\mathscr{A}$.}

P. Cartier (Orsay)

9494:

Chevalley, C. Sur la théorie de la variété de Picard. Amer. J. Math. **82** (1960), 435–490.

Let U be a complete normal variety. A divisor in this paper is a divisor which is locally principal in the usual sense. Let $G(U)$ be the group of classes (modulo linear equivalence) of divisors on U which are locally principal. Then a pair (P_0, α_0) of an abelian variety P_0 and an algebraic homomorphism α_0 of P_0 into $G(U)$ is called the Picard variety of U, if, whenever β is an algebraic homomorphism of a group variety G into $G(U)$, there is a unique homomorphism γ of G into P_0 with $\beta = \alpha_0\gamma$. We shall call this Picard variety the Picard variety of U in the strict sense. The usual Picard variety and the Picard variety in the strict sense coincide when U is nonsingular. Next, let (A_0, f_0) be a pair of an abelian variety A_0 and a morphism f_0 (everywhere defined rational map) of U into A_0. (A_0, f_0) is called the Albanese variety of U in the strict sense if the image of U by f_0 generates A_0, and if, moreover, for every morphism g of U into an abelian variety B, there is a morphism h of A_0 into B such that $g = hf_0$.

The main results of this paper are: (i) the Picard variety of U in the strict sense exists; (ii) the Picard variety of U in the strict sense is the Picard variety (in the ordinary sense) of the Albanese variety of U in the strict sense. These results are proved, approximately, in the following order. First, some results on divisors (known explicitly or implicitly) are generalized to divisors which are locally principal (special emphasis is placed on algebraic families of divisors and divisor classes). After constructing the jacobian variety J of a complete nonsingular curve, and showing that the Picard variety of J is J itself, the existence of the Picard variety of an abelian variety is proved (from the proposition that when U, U' are complete normal varieties such that there is a subjective morphism of U' into U, then U has the Picard variety in the strict sense, if U' does). Next, the existence of the Albanese variety of U in the strict sense (A_0, f_0) is given; here, A_0 turns out to be a quotient abelian variety of the usual Albanese variety A, and f_0 the composite map of the canonical map of A into A_0 and of the canonical map of U into A. Let, finally, (P_0, α_0) be the Picard variety of A_0. Then it is shown that P_0, together with the map $z \to f*(\alpha(z))$, is the Picard variety of U in the strict sense.

T. Matsusaka (Princeton, N.J.)

9495:

Nakai, Yoshikazu. Ramifications, differentials and different on algebraic varieties of higher dimensions. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **32** (1960), 391–411.

Soient V et V° deux variétés, et f un morphisme de V° sur V définis sur k algébriquement clos; on dit que f est un revêtement, si $f^{-1}(P)$ est fini pour tout $P \in V$, si V° est complète au dessus de P, et si $k(V^\circ)$ est une extension algébrique séparable de $k(V)$; dans le cas normal il revient au même de dire que V° est la normalisation de V dans $k(V^\circ)$. Soient W° une sous variété de V°, $W = f(W^\circ)$, $\mathfrak{o}^\circ$ et $\mathfrak{o}$ les anneaux locaux de W° sur V° et de W sur V, et $\mathfrak{m}^\circ$ et $\mathfrak{m}$ leurs idéaux maximaux; on dit que W° est non-ramifiée si $\mathfrak{m}^\circ = \mathfrak{m}\mathfrak{o}^\circ$ et si $\mathfrak{o}^\circ/\mathfrak{m}^\circ$ est algébrique séparable sur $\mathfrak{o}/\mathfrak{m}$. On suppose désormais V et V° normales, et on note r leur dimenaion.

Une forme différentielle ω de degré r est dite finie [resp. nulle] en un point P de V si on peut l'écrire $\omega = a dt_1 \cdots dt_r$, où a et les t_i sont éléments de l'anneau local de P [resp. et si, de plus, $a(P) = 0$]. Soient D une sous-variété de codimension 1 de V, $t_1, \cdots, t_r$ des paramètres uniformisants le long de D, $\omega = dt_1 \cdots dt_r$, et D° une composante de $f^{-1}(D)$; pour que la forme différentielle $f^*\omega$ soit nulle le long de D°, il faut et il suffit que D° soit ramifiée; le coéfficient $e(D^\circ)$ de D° dans le diviseur de $f^*\omega$ est indépendant des paramètres uniformisants $t_1, \cdots, t_r$ choisis; le diviseur $\sum e(D^\circ)D^\circ$ est appelé la différente du revêtement, et est noté $d(V^\circ/V)$.

Si P est un point simple de V, les formes différentielles de degré r finies en P forment un module libre de rang 1 sur l'anneau local de P; la réciproque est vraie. Si une forme différentielle ω est finie en un point singulier P de V, une des composantes du diviseur de ω passe par P.

Les sous-variétés ramifiées maximales de V° sont de codimension 1. On a $(f^*\omega) = f^{-1}((\omega)) + d(V^\circ/V)$ pour toute forme différentielle ω de degré r sur V. Formule de transitivité des différentes. Avec les notations ci dessus, $e(D^\circ)+1$ est égal à l'indice réduit de ramification de la valuation v° correspondant à D° sur la valuation v correspondant à D $(= f(D^\circ))$, si cet indice n'est pas multiple de la caractéristique et si $k(D^\circ)$ est séparable sur $k(D)$.

Etude des systèmes linéaires N sur une variété V tels que l'application rationnelle associée à N soit un revêtement; la différente de ce revêtement s'appelle alors le diviseur jacobien de N, et se note N_J. On a $N_J - (r+1)N \sim K$, où $r = \dim(V)$ et où K est un diviseur canonique de V. Si $|C|$ et $|C'|$ sont des systèmes linéaires donnant lieu à des revêtements, il en est de même de $|C+C'|$, et on a $|C+C'|_J = |C_J + (r+1)C'| = |C_J' + (r+1)C|$. Pour qu'une variété V admette un système linéaire donnant lieu à un revêtement, il faut et il suffit qu'elle soit projective.

P. Samuel (Clermont-Ferrand)

9496:

Igusa, Jun-ichi. Betti and Picard numbers of abstract algebraic surfaces. Proc. Nat. Acad. Sci. U.S.A. **46** (1960), 724–726.

Define the "good" Betti numbers of a nonsingular algebraic surface V by setting $B_0 = B_4 = 1$, $B_1 = B_3 =$ twice the dimension of the Picard variety of V, and by taking B_2 such that $\sum_i (-1)^i B_i =$ Euler characteristic χ of $V =$ (Zeuthen-Segre invariant of V) + 4. A proof is outlined for the relation $B_2 \geq \rho =$ Picard number of $V =$ Severi base number. The author begins by associating to any nonzero linear differential ω on V a 0-cycle $\langle\omega\rangle = \sum_{P \in V} m_P(\omega)P$ as follows: Taking uniformizing parameters t_1, t_2 at P and writing $\omega = f \cdot (h_1 dt_1 + h_2 dt_2)$, where h_1, h_2 are defined at P and have no common zero curve through P, one sets $m_P(\omega) =$ multiplicity of P in $(h_1) \cdot (h_2)$; $\langle\omega\rangle$ will be well-defined. Letting Ω be a nonzero double differential on V, define (Ω) and (ω) as usual. Then $\langle\omega\rangle + (\omega) \cdot (\Omega) - (\omega) \cdot (\omega)$ defines a rational equivalence class on V, independent of ω and Ω, whose degree is χ. As in the author's previous papers on fiber systems of jacobian varieties [Amer. J. Math. **78** (1956), 171–199, 745–760; MR **18**, 935, 936], a numerical function of f on V that gives a suitable fibering

of V into curves $\{C_u\}$ is chosen; then, using $\langle df \rangle$, χ can be computed in terms of singular points of the fibering, etc. A detailed computation involving the author's abstract vanishing cycle theory [Proc. Japan. Acad. **34** (1958), 589–593; MR **21** #50] then effects the proof.

M. Rosenlicht (Berkeley, Calif.)

LINEAR ALGEBRA

See also B9990.

9497:

Petrone, Luigi. ★Elementi di calcolo delle matrici. Serie di statistica—Teoria e applicazioni, 14. Paolo Boringhieri, Torino, 1960. vi+115 pp. L. 1200.

This is a brief and very formal first introduction into matrix algebra and some of its applications in numerical analysis. Contents: (1) Vectors (introduced as number columns or rows), linear independence, basis, invariance of dimension; (2) Systems of linear equations; (3) Matrices (formal definition and determinants); (4) Algebra of matrices (sum, product, inverse); (5) Rank of a matrix; (6) Complements on square matrices (infinite series, exponential function, norm of a matrix, Cayley-Hamilton identity); (7) Methods of numerical analysis (linear system, reduced into one with triangular matrix, iteration method, practical method to compute the inverse). For most users of matrix algebra the bare rudiments contained in this little book will not be sufficient; it will, however, have given them a good basis for the study of a more advanced work.

H. Schwerdtfeger (Montreal)

9498:

Best, Gilbert C. Two theorem tables of matrix algebra. Math. Comput. **15** (1961), 19–22.

This paper presents many theorems of matrix algebra in tabular form but does not indicate sources of the theorems nor provide proofs. Symbols are used for operators and properties. For example if k is defined to indicate commutativity, the theorem that the operators O^T and O^{-1} are commutative, formally given by $O^TO^{-1}A = O^{-1}O^TA$, is indicated in table 2 by a k at the intersection of the row representing O^T and the column representing O^{-1}, and by a k at the intersection of the row representing O^{-1} and the column representing O^T. There are 22 different operators defined and featured and 22 properties. The theorems involving operators versus properties are given in table 1 and the theorems involving operations versus operations, such as the illustration above, are given in table 2.

P. S. Dwyer (Ann Arbor, Mich.)

9499:

Grimm, C. A. A vector solution of simultaneous linear equations. Amer. Math. Monthly **67** (1960), 263–265.

Let $e_1, \cdots, e_n$ be an orthonormal base system of an n-dimensional vector space, and $V_i = v_i^r e_r$ $(i = 1, \cdots, n-1)$ the $n-1$ vectors of the space (here and below, $v_i^r e_r = \sum_{r=1}^{n} v_i^r e_r$). The vector V represented by the determinant $|c_{ij}|$, where $c_{1j} = e_j$, $c_{ij} = v_{i-1}^j$ $(i = 2, \cdots, n, j = 1, \cdots, n)$, when expanded in terms of the first row, defines the vector product of $n-1$ vectors V_i. If the determinant $|b_j^r|$ $(j, r, = 1, \cdots, n)$ of the n points $(b_j^1, \cdots, b_j^n)$ is different from zero, the $n-1$ vectors $(b_s^r - b_1^r)e_r$ $(s = 2, \cdots, n)$ are linearly independent in the hyperplane determined by the points. The main theorem reads: The coefficient vector $C = A_r e_r$ of the hyperplane $A_r x^r = M \neq 0$ $(r = 1, \cdots, n)$, determined by n points whose det $b_j^r \neq 0$, is a scalar multiple of the vector product of $n-1$ linearly independent vectors in the plane. The theorem is applied to the solution of simultaneous system of linear equations $a_j^r x_r = M \neq 0$ $(j = 1, \cdots, n)$, det $a_j^r \neq 0$. By interpreting $(a_j^1, \cdots, a_j^n)$ as n points and x_r as the components of the coefficient vector of the hyperplane determined by the points one obtains that $Nv^r = x_r$, where v^r are components of the vector product V of $n-1$ linearly independent vectors $(a_s^r - a_1^r)e_r$. The scalar coefficient N can be determined by substituting x_r in any of the equations $a_j^r x_r = M$.

J. W. Andrushkiw (Newark, N.J.)

9500:

Bauer, F. L.; Fike, C. T. Norms and exclusion theorems. Numer. Math. **2** (1960), 137–141.

For any vector norms $\|x\|$, let inf $(A) = \min_{\|x\|=1} \|Ax\|$, sup $(A) = \max_{\|x\|=1} \|Ax\|$, where A is a matrix and x a vector. The main theorem is that for any matrix B the set $[z\colon \inf(zI - B) \leq \sup(A - B)]$ contains all characteristic roots of A. When A and B are hermitian and the norm is euclidean the theorem is well known, but the general statement seems to be new. The theorem is of most interest when the norm is such that, for diagonal D, inf (D) and sup (D) are, respectively, the least and the greatest of the moduli of the diagonal elements of D.

A. S. Householder (Oak Ridge, Tenn.)

9501:

Kingman, J. F. C. On an inequality in partial averages. Quart. J. Math. Oxford Ser. (2) **12** (1961), 78–80.

If $a_{ij} \geq 0$, $p_i \geq 0$, $q_j \geq 0$, $\sum p_i = \sum q_j = 1$, and summation over suffices is denoted by replacement of the suffix by dots, the author gives a very simple proof that

$$\sum_{i,j} a_{ij} a_{i.} a_{.j} p_i q_j \geq a_{..}^3.$$

A generalisation to products of sums of $a_{ijk}\ldots$ taken over subsets of the index set $(ijk\cdots)$ is also obtained. This generalisation has important genetical applications.

P. A. P. Moran (Canberra)

9502:

Atkinson, F. V.; Watterson, G. A.; Moran, P. A. P. A matrix inequality. Quart. J. Math. Oxford Ser. (2) **11** (1960), 137–140.

Let A be an m by n matrix with non-negative elements. Let s_1, s_2, s_2' and s_3 denote the sums of the elements of A, AA^T, A^TA and AA^TA respectively. The authors prove that $mns_3 \geq s_1^3$, with equality if and only if all row sums are equal or all column sums are equal. The result is both non-trivial and suggestive. For example, the Cauchy inequality is equivalent to $ns_2 \geq s_1^2$ or to $ms_2' \geq s_1^2$. Is $2mns_3 \geq (ns_2 + ms_2')s_1$? What is the proper generalization to the sum of elements of AA^TAA^T?

K. Goldberg (Washington, D.C.)

9503:

Marcus, Marvin. Some properties and applications of doubly stochastic matrices. Amer. Math. Monthly **67** (1960), 215–221.

According to a theorem of G. Birkhoff, the set Ω_n of

n-square doubly stochastic matrices is the convex hull of m permutation matrices for $m \leq (n-1)^2+1$. It follows that a real convex (concave) function on Ω_n assumes its maximum (minimum) value for a permutation matrix. The author discusses the role played by this principle in various questions concerning the relationships among eigenvalues, singular values, and quadratic forms associated with a linear transformation. For example, let $u_1, \cdots, u_n$ be an orthonormal set of eigenvectors of a non-negative Hermitian transformation A, corresponding to the eigenvalues $0 \leq \lambda_1 \leq \lambda_2 \cdots \leq \lambda_n$. If $x_1, \cdots, x_n$ is any orthonormal set of vectors, $\alpha_j = (Ax_j, x_j)$, and λ, α are vectors with components λ_j, α_j, respectively, then $\alpha = S\lambda$, where $S = (|(x_i, u_j)|^2)$ is doubly stochastic. The relation $\prod_{j=1}^{k} \alpha_j \geq \prod_{j=1}^{k} \lambda_j$, $1 \leq k \leq n$, due to K. Fan, follows immediately from the concavity of the function

$$g(z_1, \cdots, z_n) = \Big(\prod_{j=1}^{k} z_j\Big)^{1/k}$$

for non-negative variables.

The author also discusses two conjectures of van der Waerden and Kakutani on doubly stochastic matrices.

B. N. Moyls (Vancouver, B.C.)

9504:

Ostrowski, A. M. A quantitative formulation of Sylvester's law of inertia. II. Proc. Nat. Acad. Sci. U.S.A. **46** (1960), 859–862.

In part I of this paper [same Proc. **45** (1959), 740–744; MR **22** #1588] the author had obtained inequalities connecting the characteristic roots of a hermitian matrix H with those of $S^T HS$, where S is a real square matrix. His main result is now generalized in the following manner. Let H be a hermitian $n \times n$ matrix, S a complex $n \times m$ matrix, and $K = S^* HS$. If $\pi_H(\nu_H)$ denotes the number of positive (negative) characteristic roots of H and if π_K, ν_K are defined analogously, then $\pi_K \leq \pi_H$, $\nu_K \leq \nu_H$. Furthermore, denote by $\lambda_1^+ \geq \lambda_2^+ \geq \cdots$, $\Lambda_1^+ \geq \Lambda_2^+ \geq \cdots$ the positive characteristic roots of H, K respectively; and by $\lambda_1^- \leq \lambda_2^- \leq \cdots$, $\Lambda_1^- \leq \Lambda_2^- \leq \cdots$ the negative characteristic roots of H, K respectively. Then none of the numbers Λ_i^+/λ_i^+ $(i \leq \pi_K)$, Λ_i^-/λ_i^- $(i \leq \nu_K)$ exceeds the greatest characteristic root of S^*S. Generalizations are also obtained of several other results proved in part I.

L. Mirsky (Sheffield)

9505:

Ermolaev, Yu. B. Simultaneous reduction of a pair of bilinear forms to canonical form. Dokl. Akad. Nauk SSSR **132** (1960), 257–259 (Russian); translated as Soviet Math. Dokl. **1**, 523–525.

The author considers the simultaneous reduction to canonical form of a pair of bilinear forms $x'Ay$ and $x'By$, where A and B are n-square matrices over the field K. Canonical forms are given when (i) $A' = \alpha A$, $B' = \beta B$, $\alpha^2 = \beta^2 = 1$, and K is the field of real or complex numbers; or (ii) $A' = \alpha A$, $\alpha^2 = 1$, B is Hermitian, and K is the field of complex numbers.

B. N. Moyls (Vancouver, B.C.)

9506:

Ostrowski, Alexander. Über geränderte Determinanten und bedingte Trägheitsindizes quadratischer Formen. Monatsh. Math. **64** (1960), 51–63.

The author characterizes the inertial character (i.e., signature and rank) of a real quadratic form $x'Ax$ subject to the linear restraints $Bx = 0$ (A is symmetric, B is $(k \times n)$). The results are applied to the problem of relative extrema of real functions of several variables.

H. S. A. Potter (Aberdeen)

9507:

Marcus, Marvin; Khan, N. A. On a commutator result of Taussky and Zassenhaus. Pacific J. Math. **10** (1960), 1337–1346.

Für zwei $n \times n$ Matrizen A, B sei $[A, B] = AB - BA'$, wo A' die Transponierte von A ist, ferner $[A, B]_k = [A, [A, B]_{k-1}]$ $(k = 2, 3, \cdots)$. In dem System M_n aller $n \times n$ Matrizen über einem Körper F wird durch $T(Y) = [A, Y]$ $(Y \in M_n)$ für festes $A \in M_n$ eine lineare Abbildung T erklärt, so daß $T^k(Y) = [A, Y]_k$. Verff. untersuchen für beliebiges A die Struktur des Nullraumes $\eta(T^k)$ von T^k $(k = 1, 2)$. Bezeichnet $\sigma(T^k)$ bzw. $\gamma(T^k)$ den Teilraum der in $\eta(T^k)$ enthaltenen symmetrischen bzw. schiefsymmetrischen Matrizen, so gilt im Fall char $F \neq 2$ (wegen $X = (X + X')/2 + (X - X')/2$) die direkte Zerlegung $\eta(T^k) = \sigma(T^k) + \gamma(T^k)$ $(k = 1, 2)$. Ist $E_{ij} \in M_n$ die Matrix mit einer 1 an der Stelle (i, j) und sonst überall Nullen, so hat T bez. der lexikographisch geordneten Basis $(E_{ij})_{i,j=1,\cdots,n}$ die Matrizendarstellung $T = I \otimes A - A \otimes I$, wo $\otimes$ das Kroneckerprodukt bedeutet. Mittels Übergang zur Jordanschen Normalform für A werden die Formeln

$$\dim \eta(T) = \sum_{i=1}^{p} \Big[\sum_{j=1}^{n_i} (r_{ij}^2 e_{ij} + 2 r_{ij} \sum_{k=j+1}^{n_i} r_{ik} e_{ik}) \Big],$$

$\dim \sigma(T) = \frac{1}{2} \dim \eta(T) + \frac{1}{2} \sum_{j=1}^{p} \sum_{j=1}^{n_i} r_{ij} e_{ij}$ und entsprechende Ausdrücke für $\dim \eta(T^2)$ und $\dim \sigma(T^2)$ abgeleitet. Hierbei sind $(x - \lambda_i)^{e_{ij}}$ $(j = 1, \cdots, n_i,\ e_{i1} > \cdots > e_{in_i},\ i = 1, \cdots, p)$ die verschiedenen Elementarteiler von A, und r_{ij} gibt an, wie oft $(x - \lambda_i)^{e_{ij}}$ auftritt. Insbesondere folgt daraus ein Resultat von O. Taussky und H. Zassenhaus [Pacific J. Math. **9** (1959), 893–896; MR **21** #7216] über nicht-derogatorische Matrizen.

H.-J. Hoehnke (Berlin)

9508:

Amir-Moéz, Ali R.; Davis, Chandler. Generalized Frobenius inner products. Math. Ann. **141** (1960), 107–112.

Let $\mathfrak{R}$ be a finite-dimensional complex or real *-algebra and, for $\alpha \in \mathfrak{R}$, write $\alpha \geq 0$ if α is positive semi-definite. Further, let $\mathfrak{A}$ be a left $\mathfrak{R}$-module and suppose that a function $\langle\ ,\ \rangle$ on $\mathfrak{A} \times \mathfrak{A}$ to $\mathfrak{R}$ satisfies the following conditions for all $A, B, C \in \mathfrak{A}$ and all $\alpha, \beta \in \mathfrak{R}$: (i) $\langle B, A \rangle = \langle A, B \rangle^*$. (ii) $\langle \alpha A + \beta B, C \rangle = \alpha \langle A, C \rangle + \beta \langle B, C \rangle$. (iii) $\langle A, A \rangle \geq 0$; $\langle A, A \rangle = 0$ if and only if $A = 0$. For example, $\mathfrak{R}$ can be taken as the field of complex numbers, $\mathfrak{A}$ as the set of $n \times n$ matrices over $\mathfrak{R}$, and $\langle A, B \rangle$ as the Frobenius inner product of A and B, i.e., $\langle A, B \rangle = n^{-1}$ trace (AB^*). The object of the present paper is to discuss to what extent the generalized inner product $\langle\ ,\ \rangle$ inherits the properties of the ordinary inner product on vector spaces or of the Frobenius inner product defined above. It is shown, in particular, that analogues of Bessel's inequality, of the Fourier expansion theorem, and of the Gram-Schmidt orthogonalization process can be established in the general case under consideration.

L. Mirsky (Sheffield)

9509:

Motzkin, T. S. Determinants whose elements have equal norm. Proc. Amer. Math. Soc. **11** (1960), 871–874.

Let F be a field of arbitrary characteristic and let F^* be an extension field of degree 2 over F. The conjugate of an element α of F^* is denoted by $\bar{\alpha}$, and the norm of α is defined as $\alpha\bar{\alpha}$. The following theorem is established (previously proved by the author for the case in which F was the field of real numbers):

A vanishing 3 by 3 determinant of elements of F^* of equal norms has two proportional rows or columns.

The paper also contains a number of other results about determinants of this type. *W. Ledermann* (Manchester)

ASSOCIATIVE RINGS AND ALGEBRAS

9510:

Johnson, R. E. Structure theory of faithful rings. III. Irreducible rings. Proc. Amer. Math. Soc. **11** (1960), 710–717.

The author studies in detail the irreducible components previously encountered in the direct sum representation of a ring [Trans. Amer. Math. Soc. **84** (1957), 523–544; MR **18**, 869]. For semisimple rings, these components are primitive rings. More generally, a ring R is called irreducible if there is no non-zero ideal S of R such that $0S^{-1}\neq 0$ and $S\cap 0S^{-1}=0$ and if there is no x in R such that $x^{-1}0\cap B\neq 0$ for every non-zero module B in R. Theorem: An irreducible ring R is prime if and only if R contains no non-zero nilpotent ideals. Irreducibility of a ring is related to homogeneity of the closure operation on its lattice of modules.

P. M. Whitman (Silver Spring, Md.)

9511:

Bourne, Samuel. On locally compact halfrings. Proc. Japan Acad. **36** (1960), 192–195.

A halfring is a semiring which is embeddable in a ring. A halfring H can be embedded canonically in the ring $R=H\times H/\Delta$, where Δ is the diagonal of $H\times H$, by means of the isomorphism $x\to\nu\ (x+a, a)$, where ν is the natural homomorphism $H\times H\to R$. A semisimple halfring H is said to be strongly semisimple if the above ring R is also semisimple. A topological halfring H has property F [see N. J. Rothman, Math. Ann. **139** (1960), 197–203; MR **22** #6871] if, whenever i, $x\in H$, and V_i is an open neighborhood of i, there exists an open neighborhood U_x of x such that $x+i\in\bigcap[V_i+x'|x'\in U_x]$. A topological halfring H is right-bounded [see I. Šafarevič, Dokl. Akad. Nauk SSSR **40** (1943), 149–151; MR **6**, 164] if for any open neighborhood U of zero there exists an open neighborhood V of zero such that $V\cdot H$ is contained in U. H is bounded when it is both right-bounded and left-bounded. The author proves: (i) if H has property F, then the homomorphism ν is an open mapping; (ii) a locally compact bounded strongly semisimple halfring with property F is embeddable in a locally compact bounded semisimple ring. *P. Abellanas* (Madrid)

9512:

Herstein, I. N. Power maps in rings. Michigan Math. J. **8** (1961), 29–32.

Let R be any associative ring and, for any fixed integer $n>1$, consider the two identities (*) $(xy)^n=x^ny^n$, (**) $(x+y)^n=x^n+y^n$ (for all x, y in R). The author proves that, given either (*) or (**), every commutator $ab-ba$ in R is nilpotent and the nilpotent elements of R form an ideal. Further, if both (*) and (**) hold, i.e., if the map $x\to x^n$ is homomorphic, and if this map is onto R, then R must actually be commutative; this generalizes Jacobson's result [Ann. of Math. (2) **46** (1945), 695–707; MR **7**, 238] that a ring in which $x^n=x$ for all x must be commutative. The author's proofs are brief and elegant, but depend on relatively deep results of Amitsur [Proc. Amer. Math. Soc. **8** (1957), 649–656; MR **19**, 384], Cartan, Brauer and Hua [H. Cartan, Ann. Sci. École Norm. Sup. (3) **64** (1947), 59–77; MR **9**, 325; I. Barsotti, Rend. Mat. e Appl. (5) **7** (1948), 1–30; MR **10**, 179; R. Brauer, Bull. Amer. Math. Soc. **55** (1949), 619–620; MR **10**, 676; L.-K. Hua, Proc. Nat. Acad. Sci. U.S.A. **35** (1949), 533–537; MR **11**, 155], and Kaplansky [Bull. Amer. Math. Soc. **54** (1948), 575–580; Canad. J. Math. **3** (1951), 290–292; MR **10**, 7; **13**, 101]. *M. P. Drazin* (Baltimore, Md.)

9513:

Elizarov, V. P. Rings of quotients for associative rings. Izv. Akad. Nauk SSSR. Ser. Mat. **24** (1960), 153–170. (Russian)

Let R be an associative ring and S a subset of R which is closed under multiplication but does not contain zero. If R is commutative, then it is well known that there always exists a ring of quotients for R with respect to S. The author considers the general case and defines a generalized ring of quotients for R with respect to S which reduces to the usual ring when R is commutative, and to a ring constructed by Asano [J. Math. Soc. Japan **1** (1949), 73–78; MR **11**, 154] when S does not contain divisors of zero of R. First define a homomorphism φ of R into another ring R' to be S-reducing if it satisfies the conditions: (1) all elements of $\varphi(S)$ have 2-sided inverses in R'; (2) if $x\in R'$, there exist elements $\varphi(s)\in\varphi(S)$ and $\varphi(r)\in\varphi(R)$ such that $x=[\varphi(s)]^{-1}\varphi(r)$. Next define a two-sided ideal I in R to be S-prime if it satisfies the conditions: (1) I contains no elements of S; (2) if $r\in R$, $s\in S$, and $rs\in I$ or $sr\in I$, then $r\in I$; (3) for arbitrary $r\in R$ and $s\in S$, there exist elements $r_1\in R$ and $s_1\in S$ such that $s_1r-r_1s\in I$. The author shows that a two-sided ideal I in R is S-prime if and only if it is the kernel of some S-reducing homomorphism of R. Finally, a ring $R_{(S)}$ is called a generalized "left" ring of quotients for R with respect to S if it satisfies the following conditions: (a) there exists an S-reducing homomorphism of R into $R_{(S)}$; (b) if φ is an S-reducing homomorphism of R into a ring R', there exists a homomorphism ψ of $R_{(S)}$ into R' such that $\psi(\varphi(R))=\varphi(R)$. The author proves that R admits a generalized left ring of quotients $R_{(S)}$ if and only if the intersection of all S-prime ideals in R is also S-prime. Similar definitions and results hold for "right" rings of quotients for R with respect to S. The author studies these rings of quotients, obtaining, for example, a uniqueness theorem, isomorphism properties, properties of ideals, and existence of rings of quotients for factor rings of R. *C. E. Rickart* (New Haven, Conn.)

9514:

Cohn, P. M. On a generalization of the Euclidean algorithm. Proc. Cambridge Philos. Soc. **57** (1961), 18–30.

Soit R un anneau (non nécéssairement commutatif). Un degré d sur R est une fonction à valeurs entières ≥ 0 telle que $-d$ soit une valuation. On dit qu'un degré d est euclidien si, pour tous $a, b \in R$ tels que $d(a) \geq d(b) \geq 0$, il existe $c \in R$ tel que $d(a-bc) < d(a)$; on peut alors écrire $a = bq + r$ avec $d(r) < d(b)$, et tout idéal à droite de R est principal. En notant k le corps formé par les éléments de degré 0, R est alors un anneau de polynômes non commutatifs $\sum_n X^n a_n$ à une variable sur k (où on a la règle de commutation $aX = Xs(a) + D(a)$, s étant un endomorphisme et D une s-dérivation de k); réciproquement, le degré au sens usuel dans un tel anneau de polynômes non-commutatifs est un degré euclidien. On dit qu'un degré d sur R est quasi-euclidien si, pour tous éléments $a_1, \cdots, a_r$, $b_1, \cdots, b_r$ de R vérifiant $d(a_1 b_1) = \cdots = d(a_r b_r) > d(\sum_{i=1}^{r} a_i b_i)$ et $d(a_1) \geq d(a_i)$, il existe $c_2, \cdots, c_r$ dans R tels que $d(a_1 - \sum_{i=2}^{r} a_i c_i) < d(a_1)$ et que $d(a_i c_i) \leq d(a_1)$. Lorsque deux éléments quelconques non nuls de R ont un commun multiple à droite, tout degré quasi-euclidien sur R est euclidien. Dans un anneau R admettant un degré quasi-euclidien, tout idéal à droite est un R-module libre. Etude de l'engendrement de tels anneaux. Une algèbre associative libre n'admet pas de degré quasi-euclidien.

P. Samuel (Clermont-Ferrand)

9515:

Northcott, D. G. Prime ideals and Dedekind orders. Proc. London Math. Soc. (3) **10** (1960), 480–496.

Soit R un anneau noethérien intègre dont tout idéal premier non nul est maximal. Pour tout idéal $\mathfrak{a} \neq 0$ de R, on a $l(R/\mathfrak{a}^n) = e(\mathfrak{a})n - r(\mathfrak{a})$ pour n grand, où $e(\mathfrak{a})$ et $r(\mathfrak{a})$ sont des entiers. Lorsque $\mathfrak{a}$ est premier, les relations "$e(\mathfrak{a}) = 1$", "$r(\mathfrak{a}) = 0$" et "$R_{\mathfrak{a}}$ est un anneau de valuation discrète" sont équivalentes. Pour qu'un idéal $\mathfrak{a} \neq 0$ soit inversible, il faut et il suffit que $r(\mathfrak{a}) = 0$, ou encore que $e(\mathfrak{a}) = l(R/\mathfrak{a})$. On dit que R est un ordre de Dedekind si, de plus, les idéaux premiers singuliers $\mathfrak{p}$ de R (c'est à dire non inversibles, ou encore tels que $e(\mathfrak{p}) \geq 2$) sont en nombre fini. Notant alors $\mathfrak{b}$ le produit de ceux ci, le sous anneau $R^{(1)} = \bigcup_{n=1}^{\infty} (\mathfrak{b}^n : \mathfrak{b}^n)$ du corps des fractions de R est un R-module de type fini, appelé l'anneau de réduction de R; tous les idéaux premiers de R sont inversibles dans $R^{(1)}$, et $R^{(1)}$ est un ordre de Dedekind. Soient $R^{(2)}$ l'anneau de réduction de $R^{(1)}$, $R^{(3)}$ celui de $R^{(2)}$, etc.; alors la réunion I des $R^{(n)}$ est la clôture intégrale de R, et est un anneau de Dedekind classique. Pour que I soit un R-module de type fini, il faut et il suffit que la suite $(R^{(n)})$ soit stationnaire. Dans le cas géométrique, cette construction correspond à la réduction des singularités d'une courbe au moyen d'une suite de transformations quadratiques.

P. Samuel (Clermont-Ferrand)

9516:

Dickson, L. E. ★Linear algebras. Reprinting of Cambridge Tracts in Mathematics and Mathematical Physics, No. 16. Hafner Publishing Co., New York, 1960. viii + 73 pp. $3.00.

A reprint of the 1914 edition [Cambridge Univ. Press, London].

9517:

Kodama, Tetsuo. Note on the commutator group of normal simple algebra. Mem. Fac. Sci. Kyushu Univ. Ser. A **14** (1960), 98–103.

The author [same Mem. **10** (1956), 141–149; MR **19**, 631] previously proved Tannaka's conjecture that if A is a normal simple algebra over a field k of characteristic 0 then every element of A with reduced norm 1 belongs to the commutator subgroup of the multiplicative group of A, following the solution in special cases by Nakayama and Matsushima [Proc. Imp. Acad. Tokyo **19** (1953), 622–628; MR **7**, 238], and S. Wang [Amer. J. Math. **72** (1950), 323–324; MR **11**, 577]. The present paper proves that the same holds also when k is an infinite field of prime characteristic. In reducing the problem to the case of a division algebra by Dieudonné's theorem, the proof makes much use of lemmas proved in the author's previous paper [loc. cit.].

T. Nakayama (Nagoya)

9518:

Nagata, Masayoshi. Some remarks on prime divisors. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **33** (1960/61), 297–299.

Soient R un anneau commutatif, $\mathfrak{a}$ un idéal de R, et $\mathfrak{p}$ un idéal premier contenant $\mathfrak{a}$. On dit que $\mathfrak{p}$ est un diviseur premier de $\mathfrak{a}$ si $\mathfrak{p}/\mathfrak{a}$ est formé de diviseurs de zéro. Construction d'un anneau (non noethérien) R tel que: (1) il existe un idéal $\mathfrak{a}$ de R et un diviseur premier $\mathfrak{p}$ de $\mathfrak{a}$ tels que $\mathfrak{p}R_{\mathfrak{p}}$ ne soit pas un diviseur premier de $\mathfrak{a}R_{\mathfrak{p}}$; (2) il existe un idéal $\mathfrak{b}$ de R, égal à sa racine, et admettant des diviseurs premiers immergés. Plus généralement, l'auteur construit un anneau T, égal à son anneau total de fractions, et dont la famille des T_m (m maximal) est donnée à l'avance.

P. Samuel (Clermont-Ferrand)

9519:

Goldberg, K. Generating functions for formal power series in noncommuting variables. Proc. Amer. Math. Soc. **11** (1960), 988–991.

L'auteur donne un algorithme pour le calcul des coefficients d'une série formelle $A(f_1(x_1), \cdots, f_n(x_n))$, où $A(x_1, \cdots, x_n)$ est une série formelle donnée par rapport à des indéterminées non permutables x_i, et chacune des $f_i(x_i)$ est une série formelle donnée sans terme constant. La méthode consiste à grouper dans A les termes en $x_{s_1}^{t_1} \cdots x_{s_m}^{t_m}$ pour une suite arbitraire donnée $(s_1, \cdots, s_m)$ de nombres $\leq n$ et à associer à cette série partielle la série formelle par rapport à des indéterminées permutables $z_1, \cdots, z_m$ obtenue en substituant z_i à x_{s_i}; la substitution de $f_i(x_i)$ à x_i correspond alors à la substitution de $f_{s_i}(z_i)$ à z_i.

J. Dieudonné (Paris)

9520:

Warner, Seth. Compact noetherian rings. Math. Ann. **141** (1960), 161–170.

The author proves the following theorems: (a) A compact noetherian ring A is semi-local or local according as A does or does not possess zero divisors; and its given topology is the natural topology of A. (b) Let A_S be a ring of quotients of a semi-local ring A. Then, A_S is a topological ring inducing on A its natural topology and for which A is open, if and only if every proper prime

ideal of A intersecting S is maximal. The topology of A_S, when it exists, is unique, the set of elements invertible in A_S is open, and $x \to x^{-1}$ is continuous on that set. (c) Let A be a semi-local ring, $B = A[[X_1, \cdots, X_r]]$ the ring of formal power series in r indeterminates over A. (1) The topology induced on A by the natural topology of B is the natural topology of A. (2) A is closed in B. (3) B is complete if and only if A is complete. (4) B is compact, totally bounded, or separable, respectively, if and only if A is.

P. Abellanas (Madrid)

9521:

Rees, D. a-transforms of local rings and a theorem on multiplicities of ideals. Proc. Cambridge Philos. Soc. **57** (1961), 8–17.

If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of a noetherian ring A such that $\mathfrak{a}^{r+1} = \mathfrak{a}^r\mathfrak{b}$ when r is a sufficiently large integer, and if $\mathfrak{a} \supseteq \mathfrak{b}$, then $\mathfrak{b}$ is called a reduction of $\mathfrak{a}$. The author has shown elsewhere [D. G. Northcott and D. Rees, same Proc. **50** (1954), 145–158; MR **15**, 596], that if A is a local ring with maximal ideal $\mathfrak{m}$ and if $\mathfrak{a}$ is $\mathfrak{m}$-primary then so is $\mathfrak{b}$, and moreover $e(\mathfrak{a}) = e(\mathfrak{b})$, where e denotes the multiplicity in the sense of P. Samuel. In the present paper, a local ring is said to be level when all the minimal prime ideals of its completion have the same dimension d, and it is shown that when A is level and $\mathfrak{a}$ and $\mathfrak{b}$ are $\mathfrak{m}$-primary, the conditions $\mathfrak{a} \supseteq \mathfrak{b}$, $e(\mathfrak{a}) = e(\mathfrak{b})$ imply that $\mathfrak{b}$ is a reduction of $\mathfrak{a}$. In the course of the proof some interesting information about the $\mathfrak{a}$-transform or Rees-ring R associated with $\mathfrak{a}$ is obtained. Let $\mathfrak{p}$ be a maximal relevant prime ideal of R and let Q be the set of all quotients x/y, where x and y are elements of R of the same degree and $y \notin \mathfrak{p}$. Then Q is a local ring that dominates A and has a residue field K that is a finite algebraic extension of the residue field k of A. If $\mathfrak{q}$ is primary for the maximal ideal of Q, a sequence $\mathfrak{q}_1, \mathfrak{q}_2, \cdots$ of A-ideals can be associated with $\mathfrak{q}$ such that $\mathfrak{q}_n \supset \mathfrak{q}_{n+1}$, $\mathfrak{q}_n \subset \mathfrak{a}^n$ and $\mathfrak{q}_{n+1} = \mathfrak{a} \cdot \mathfrak{q}_n$ for large n. It is shown that the dimension of Q is at most d, that $e(\mathfrak{q}_n) = e(\mathfrak{a}^n)$ for large n when $\dim Q < d$, and that $[K : k]e(\mathfrak{q}) = e(\mathfrak{q}_n) - e(\mathfrak{a}^n)$ for large n when $\dim Q = d$.

H. T. Muhly (Iowa City, Iowa)

NON-ASSOCIATIVE RINGS AND ALGEBRAS

See also 9448.

9522:

Lyndon, R. C. Burnside groups and Engel rings. Proc. Sympos. Pure Math., Vol. 1, pp. 4–14. American Mathematical Society, Providence, R.I., 1959.

It has long been known that the Lie ring $\bar{L}$ induced in the Magnus representation of a discrete group G of exponent p ($u^p = 1$ for all $u \in G$) necessarily has characteristic p and satisfies the $(p-1)$st Engel condition: $[U, V^{p-1}] = [[\cdots[U, V], \cdots], V] = 0$. What is not clear is whether or not $\bar{L}$ may be subject to any identities other than linear consequences of these. This paper re-examines an attack on such problems first developed by O. Grün [J. Reine Angew. Math. **182** (1940), 158–177; MR **2**, 212]. The "Grün identities", as rederived and extended here, give relations among certain symmetrized elements of that ideal P in the free Lie ring L of characteristic p for which $L/P = \bar{L}$ is the Lie ring induced from the Burnside group. This presentation is more perspicuous than Grün's original discussion, and somewhat improves on the earlier results as well as amplifying them with various tactical comments of independent interest. Nevertheless, the author says he is "far from proving that the identities of Grün are consequences of the Engel condition, or even of recovering the results of Sanov and Kostrikin [I. N. Sanov, Izv. Akad. Nauk SSSR Ser. Mat. **16** (1952), 23–58; MR **13**, 721; A. I. Kostrikin, ibid. **21** (1957), 289–310; MR **20** #898]."

R. L. Davis (Chapel Hill, N.C.)

HOMOLOGICAL ALGEBRA

See also 9448.

9523:

Northcott, D. G. ★An introduction to homological algebra. Cambridge University Press, New York, 1960. xii + 282 pp. $8.00.

The author has written this book in order to acquaint the student of mathematics with the ideas and methods of homological algebra. He has assumed that the reader is familiar with the notions of group, ring, and field, but otherwise the presentation is self-contained.

Chapters 1–6 develop the theory of derived functors, and the remaining chapters are devoted to some applications. In particular, there is a chapter dealing with noetherian rings and modules, and the homological characterization of regular local rings. Another chapter discusses the homology and cohomology theory of groups and monoids, and some attention is given to finite groups.

There are two inferences that can be drawn from the author's having written this book. One is that the book *Homological algebra* by H. Cartan and S. Eilenberg [Princeton Univ. Press, Princeton, N.J., 1956; MR **17**, 1040] is too difficult for the beginner to read, and the other is that some of the recent applications of homological techniques to commutative algebra should be presented to the student in order to stimulate his interest in homology theory. Although it is generally conceded that the Cartan-Eilenberg book is more encyclopedic than inspiring, it is not clear that the first six chapters of the book under review remedy this deficiency. Although a general theory of categories and functors is discussed (chapter 3), the author later restricts himself exclusively to categories of modules and to the functors, tensor product and Hom. For all but one of the applications considered in this book, these are the only categories and functors that are needed, so that it might have been more instructive to the reader if inessential generalities had been omitted. It seems that an attempt to give a watered-down version of Cartan-Eilenberg avoids none of the dryness of that work, and provides even less inspiration.

As for the problem of presenting stimulating applications, on the one hand it is difficult to believe that the reader of this book, who is assumed to know so little algebra, can appreciate the applications of homology to the theory of local rings. On the other hand, for instance, the failure to point out that the canonical complex constructed over the residue field of a local ring has for its components the exterior products of a finitely generated free module will probably make it difficult for the reader

to fully understand some of the more recent work, using the Koszul complex in dealing with multiplicity theory and codimension.

Another criticism of this book is that the presentation of the material is neither new nor different. As mentioned earlier, the development of derived functors follows pretty much the line of development of Cartan-Eilenberg. In flirting with the notion of category, the author fails to point out that his diagrams are covariant functors and that his translations of diagrams are natural transformations of functors. In presenting the recent applications to noetherian ring theory, there seems to be little attempt to be selective about the proofs. For example, for Auslander's important theorem about the global dimension of noetherian rings being determined by the cyclic modules, the original lengthy proof is given, rather than the two- or three-line proof that is now fairly standard. The chapter on cohomology theory of groups goes no further than showing the existence of complete resolutions for finite groups, and gives no inkling as to how it comes up in class field theory.

Of course, all the foregoing remarks pertain more to the question of the need for such a book as this, than to an evaluation of the book on its own merits. Since the level of the book is supposed to be quite elementary, it might have been useful to have included some exercises. However, the author does leave some proofs to the reader and these (together with the exercises in Cartan-Eilenberg) should serve to give more facility with homological technique. It should be pointed out that the author has used his talent for expository writing, and the book is very clear and easy to read.

D. Buchsbaum (Princeton, N.J.)

9524:

Douglas, A. J. A homological characterization of certain Abelian groups. Proc. Cambridge Philos. Soc. **57** (1961), 256–264.

The author proves that if G is an abelian group, then G is torsion-free of rank p if and only if the homology groups $H_n(G, A)$ vanish for all $n>p$ and all G-modules A. Standard homological techniques (e.g., Tor commutes with direct limits) provide a proof by reducing to the elementary cases where G is finitely generated or a rational vector space. *D. Zelinsky* (Evanston, Ill.)

9525:

Isbell, J. R. Natural sums and direct decompositions. Duke Math. J. **27** (1960), 507–512.

The author defines a notion of addition of homomorphisms (natural sum) in categories possessing zero maps and finite free ($\vee$) and direct ($\times$) products for which the natural map $G \vee G \to G \times G$ is epimorphic. For the category of groups his definition reduces to summation of homomorphisms with commuting values, and for additive categories to the usual addition of homomorphisms. He defines additive direct-product decompositions (projection endomorphisms summable to an identity map), and proves these to coincide with the usual category-theoretical product-decompositions.

The author discusses summation of infinite families of homomorphisms, employing the notion of weak product = image of the (infinite) free product in the (infinite) direct product.

The author compares his treatment with that of Kuroš [Trudy Moskov. Mat. Obšč. **8** (1959), 391–412; MR **21** #3365]. Cf. also Kuroš, Livšic and Šul'gelfer [#9526].

J. M. Beck (New York)

9526:

Kuroš, A. G.; Livšic, A. H.; Šul'gelfer, E. G. Foundations of the theory of categories. Uspehi Mat. Nauk **15** (1960), no. 6 (96), 3–52. (Russian)

Extravagantly detailed reference-paper on the elements of categories and functors (initial definitions, monomorphisms, kernels, images, operator-objects, products, ...), with attention chiefly paid to the non-additive case (categories of arbitrary groups, of rings, of universal algebras, ...). Unfamiliar may be the authors' definition of the discrete direct product as the canonical image of the free product in the direct, as well as their proposal to define the same by means of projection-endomorphisms adding up to identity maps, in categories with "partial summation of mappings". [Compare Isbell, #9525.]

{One notices that the term "subcategory", though of course properly defined, is here often wrongly applied. For example, the authors in effect consider that the functor (underlying set): groups→sets exhibits the category of groups as a subcategory of that of sets.}

J. M. Beck (New York)

9527:

Ishikawa, Takeshi. A theorem on flat couples. Proc. Japan Acad. **36** (1960), 389–391.

A commutative ring R such that every finitely generated ideal is projective is called semi-hereditary. The author shows that if R is semi-hereditary and R' is an integral extension ring of R, then R'/R (whence also R') is a flat R-module; that is, the couple (R', R) is flat in the sense of Serre [Ann. Inst. Fourier Grenoble **6** (1955/56), 1–42; MR **18**, 511]. The proof proceeds by first showing that a semi-hereditary ring is integrally closed in its full ring of quotients. Then it is shown that R is integrally closed if and only if R'/R is R-torsion-free for every integral extension ring of R. Here "torsion-free" is to be interpreted to mean that no non-zero-divisor annihilates. The final step is an application of an unpublished result of Endo's which shows that torsion-free modules over semi-hereditary rings are flat. This last result is a generalization of a theorem of Hattori [J. Math. Soc. Japan **9** (1957), 381–385; MR **20** #854b], who treated the case of an integral domain. *A. Rosenberg* (Berkeley, Calif.)

9528:

Watts, Charles E. Intrinsic characterizations of some additive functors. Proc. Amer. Math. Soc. **11** (1960), 5–8.

Soient Λ et Γ deux anneaux, ${}_\Lambda\mathfrak{M}$ (resp. ${}_\Gamma\mathfrak{M}$) la catégorie des Λ-modules à gauche (resp. des Γ-modules à gauche); on note ${}_\Lambda\mathfrak{M}_\Gamma$ la catégorie des $(\Lambda-\Gamma)$-bimodules (Λ-modules à gauche et Γ-modules à droite); de même pour ${}_{\Lambda,\Gamma}\mathfrak{M}$ et ${}_\Gamma\mathfrak{M}_\Lambda$.

Théorème 1: Soit T un foncteur covariant additif de ${}_\Lambda\mathfrak{M}$ dans ${}_\Gamma\mathfrak{M}$, et $C=T(\Lambda)$. Alors $C \in {}_\Gamma\mathfrak{M}_\Lambda$, on a un morphisme de foncteurs $\varphi_X: C \otimes {}_\Lambda X \to T(X)$ (pour $X \in {}_\Lambda\mathfrak{M}$); si en outre T est exact à droite et commute aux sommes directes (infinies), φ_X est un isomorphisme pour tout X.

Théorème 2: Soit T un foncteur contravariant additif de

${}_\Lambda\mathfrak{M}$ dans ${}_\Gamma\mathfrak{M}$, et $C=T(\Lambda)$. Alors $C\in{}_\Lambda\mathfrak{M}_\Gamma$, on a un morphisme de foncteurs φ_X: $T(X)\to\mathrm{Hom}_\Lambda(X,C)$ (pour $X\in{}_\Lambda\mathfrak{M}$); si en outre T est exact à gauche et transforme les sommes directes en produits, φ_X est un isomorphisme pour tout X.

Théorème 3: Soit T un foncteur covariant additif de ${}_\Lambda\mathfrak{M}$ dans ${}_\Gamma\mathfrak{M}$; si T est exact à gauche et commute avec les produits, il existe $C\in{}_\Lambda\mathfrak{M}_\Gamma$ et un isomorphisme de foncteurs φ_X: $\mathrm{Hom}_\Lambda(C,X)\to T(X)$ (pour $X\in{}_\Lambda\mathfrak{M}$).

Les théorèmes 1 et 2 admettent des variantes lorsque A est noethérien à gauche et qu'on remplace ${}_\Lambda\mathfrak{M}$ par la catégorie des Λ-modules à gauche de type fini.

{N.B. Le théorème 3 n'est énoncé et démontré que lorsque $\Gamma=Z$, anneau des entiers naturels. Mais le cas général est une conséquence facile de ce cas particulier: sur C, on définit une structure de Γ-module à droite $(c,\gamma)\to c\cdot\gamma$, en posant $c\cdot\gamma=(\theta(\gamma)\cdot i)(c)$, où $\theta(\gamma)$ désigne l'endomorphisme de $T(C)=\mathrm{Hom}_\Lambda(C,C)$ défini par la structure de Γ-module à gauche de $T(C)$, et i désigne l'élément identique de $\mathrm{Hom}_\Lambda(C,C)$.} *H. Cartan* (Paris)

GROUPS AND GENERALIZATIONS

See also 9524, 9959, 9971, B10212.

9529:

Marchionna Tibiletti, Cesarina. Prodotti sghembi di insiemi e strutture di gruppo collegate. Ann. Univ. Ferrara. Sez. VII (N.S.) **8** (1958/59), 43–60. (French summary)

The author gives a further generalisation of her results on the structure of Rédei's skew product [see #9530 and Univ. e Politec. Torino. Rend. Sem. Mat. **17** (1957/58), 209–221; MR **21** #4177]. Suppose that G and Γ are sets with elements $a, b, \cdots$ and $\alpha, \beta, \cdots$, respectively (in all the preceding work on skew products G and Γ were assumed to be groups). Suppose that in their cartesian product a binary composition is given by the rule $(a,\alpha)(b,\beta)=(f(a,\alpha;b,\beta),\varphi(a,\alpha;b,\beta))$, where the single-valued functions f and φ have their values in G and Γ, respectively. Under what conditions is this skew product $G\circ\Gamma$ of the sets G and Γ a group? Under certain additional weak assumptions the author obtains the following necessary conditions: $G\circ\Gamma$ contains certain subgroups, G and Γ or certain subsets of them are groups, and the formation of $G\circ\Gamma$ can be decomposed into products of permutable groups and of Schreier extensions.

K. A. Hirsch (St. Louis, Mo.)

9530:

Marchionna Tibiletti, Cesarina. Una scomposizione di un più generale prodotto sghembo di gruppi. Univ. e Politec. Torino. Rend. Sem. Mat. **18** (1958/59), 77–89.

The author proves that every skew product in the sense of Rédei can be analyzed into two ordinary Schreier extensions and a product of two permutable subgroups with a known intersection. This result completes a series of investigations initiated by Rédei and continued by Kochendörffer, Rühs and the present author [Kochendörffer, J. Reine Angew. Math. **192** (1953), 96–101; MR **15**, 852; Marchionna Tibiletti, Univ. e Politec. Torino. Rend. Sem. Mat. **17** (1957/58), 209–221; MR **21** #4177; Rédei, Comment. Math. Helv. **20** (1947), 225–264; J. Reine Angew. Math. **188** (1950), 201–227; MR **9**, 131; **14**, 13; F. Rühs, ibid. **198** (1957), 81–86; **200** (1958), 99–111; MR **20** #900; **21** #1333].

K. A. Hirsch (St. Louis, Mo.)

9531:

Gorčakov, Yu. M. Primitively factorizable groups. Dokl. Akad. Nauk SSSR **131** (1960), 1246–1248 (Russian); translated as Soviet Math. Dokl. **1**, 412–415.

The classical results of P. Hall [J. London Math. Soc. **12** (1937), 201–204] on finite completely factorizable groups (i.e., those in which every subgroup is complemented) have been extended to infinite groups by N. V. Černikova (Baeva) [Dokl. Akad. Nauk SSSR **92** (1953), 877–880; Mat. Sb. (N.S.) **39** (**81**) (1956), 273–292; MR **15**, 503; **18**, 639]. The present author treats the case of groups G in which every subgroup of prime order p is complemented, where p belongs to a given set π of primes. He calls such groups primitively π-factorizable, and if π contains all the primes that divide the orders of the elements of G, primitively factorizable. A group A is called completely p-primitive if it is a completely factorizable subgroup of the holomorph of P and contains P, where P is of order p. G is called completely p-approximable if for every element of G of order p, G has a normal subgroup, not containing that element, whose factor group is completely p-primitive. If this is so for every $p\in\pi$, then G is completely π-approximable. Finally, if G has for every non-trivial element a normal subgroup, not containing that element, whose factor group is completely primitive, then G is called completely approximable.

Among the results announced by the author without proof are the following. An infinite group G is completely approximable if and only if it is a subgroup of the cartesian product of completely primitive groups. A periodic completely approximable group G can be embedded in the direct product of completely primitive groups if and only if it is locally (finite + normal). In particular, a finite group G is completely approximable if and only if it is a subgroup of the direct product of a finite number of primitive groups. A group G is completely π-approximable if and only if it is an extension of a group not containing elements of order $p\in\pi$ by a subgroup of the cartesian product of completely p-primitive groups, $p\in\pi$. An infinite group is primitively p-factorizable if and only if it is an extension of a group not containing elements of order p by a subgroup of the cartesian product of permutation groups on p symbols. Let π_n be the set of the first n primes. A periodic group G is primitively π_n-factorizable if and only if it is completely π_n-approximable. The author shows by an example, with $n=5$, that the condition of periodicity cannot be dropped. A group G is primitively π-factorizable if and only if it is locally primitively π-factorizable. The following classes of groups coincide: periodic primitively factorizable groups; periodic completely approximable groups; periodic subgroups of cartesian products of completely primitive groups; locally completely factorizable groups; periodic groups having an invariant system with cyclic factors and elementary abelian Sylow p-subgroups. *K. A. Hirsch* (Boulder, Colo.)

9532:

Conrad, P. F.; Clifford, A. H. Lattice-ordered groups having at most two disjoint elements. Proc. Glasgow Math. Assoc. **4**, 111–113 (1960).

Two elements a and b of a lattice-ordered group (l-group) L are called disjoint if $a>0$, $b>0$ and $a \wedge b=0$. The authors characterize l-groups which contain two but not more than two mutually disjoint elements. Let A and B be linearly ordered groups (o-groups) and let $A \oplus B$ be the cardinal sum of A and B. An l-group L containing $A \oplus B$ as a convex normal subgroup is called a lexico-extension of $A \oplus B$ if each $a \in L$, not belonging to $A \oplus B$, satisfies the relation $a \geqq b$ for every $b \in A \oplus B$. Theorem: An l-group which contains two but not more than two mutually disjoint elements is a lexico-extension of the cardinal sum $A \oplus B$ of two o-subgroups A and B of L by an o-group C. *Ph. Dwinger* (Lafayette, Ind.)

9533:

Boyer, D. L. A note on a problem of Fuchs. Pacific J. Math. **10** (1960), 1147.

Problem 3 in the reviewer's book *Abelian groups* [Publ. House of the Hungarian Acad. Sci., Budapest, 1958; MR **21** #5672] is solved for nondenumerable abelian groups. Let G be such a group of cardinality m, and $\mathfrak{P}$ the set of all pure subgroups of cardinality m in G. Then $\mathfrak{P}$ is of power 2^m. *L. Fuchs* (Budapest)

9534:

Baumslag, Gilbert; Blackburn, Norman. Groups with cyclic upper central factors. Proc. London Math. Soc. (3) **10** (1960), 531–544.

The authors are concerned with groups G whose upper central series $1=Z_0<Z_1<Z_2<\cdots$ extends non-trivially to at least the first limit ordinal ω and for which each Z_n/Z_{n+1} is cyclic. Their main result is that if G is a p-group with G/Z_ω abelian and Z_n/Z_{n-1} of order p, then G/Z_ω is in fact cyclic of order p^f; say, Z_ω is the direct product of $p^{f-1}(p-1)$ groups of type p^∞, and for each f there are precisely two such groups. By way of contrast they show that the set of non-isomorphic groups G for which G/Z_ω and all Z_n/Z_{n-1} are infinite cyclic has the cardinal of the continuum. Among the results that are proved on the way we mention the following: If all Z_n/Z_{n-1} are cyclic of a fixed order d, then $[G, Z_n]=Z_{n-1}$, but $[G, Z_\omega]=Z_\omega$; if all the factors are finite cyclic, but not of fixed order, then Z_ω is at least nilpotent. But the authors show by a fine example that in this case Z_ω need not be abelian. If the upper central series reaches G at the suffix γ, $Z_\gamma=G$, and if for each ordinal $\alpha<\gamma$, $Z_{\alpha+1}/Z_\alpha$ is finite cyclic of fixed order d, then $\gamma=\omega+1$ and Z_ω is the direct product of groups of type p^∞.

K. A. Hirsch (Boulder, Colo.)

9535:

Cohen, E. On the average number of direct factors of a finite abelian group. Acta Arith. **6** (1960), 159–173.

Let G be an abelian group. $\tau(G)$ denotes the number of direct factors of G and $t(G)$ denotes the number of direct decompositions of G into two relatively prime direct factors (two factors H_1 and H_2 of G are called relatively prime if the identity is the only common direct factor of H_1 and H_2).

The author proves by elementary methods that

$$(1)\quad T(x)=\sum{}'\tau(G) = \alpha' x(\log x+2\gamma-1)+\beta' x+O(x^{1/2}(\log x)^2)$$

and

$$(2)\quad T^*(x)=\sum{}'t(G) = c_1 x(\log x+2\gamma-1)+c_2 x+O(x^{1/2}\log x).$$

The dash in the summation indicates that the summation is extended over all the abelian groups of order not exceeding x. The constants in (1) and (2) are defined as follows: γ is Euler's constant. Put $Z_K(s)=\prod_{i=K}^{\infty}\zeta(is)$ ($\zeta(s)$ is Riemann's zeta function), $H(s)=Z_2{}^2(s)/Z_1(s)$, $c_1=H(1)$, $c_2=H'(1)$, $Z_2{}^2(s)=L(s)$, $\alpha'=L(1)$, $\beta'=L'(1)$.

P. Erdös (Budapest)

9536:

Levine, Norman. The anticenter of a group. Amer. Math. Monthly **67** (1960), 61–63.

Results on the set of those elements of a group which commute only with their own powers (also with fractionary exponents).

9537:

Howarth, J. C. Some automorphisms of finite nilpotent groups. Proc. Glasgow Math. Assoc. **4**, 204–207 (1960).

Soit G un groupe et $G=G_1 \supseteq G_2 \supseteq \cdots$ sa série centrale descendante ($G_i=[G_{i-1}, G]$ pour $i>1$). Si G_k est abélien et $a_1, \cdots, a_m$ sont des éléments de G_{k-1} l'application θ: $g\theta=g[g, a_1]\cdots[g, a_m]$ est un endomorphisme de G. Le produit de deux endomorphismes de ce type est encore un endomorphisme du même type. Si G est un p-groupe fini, ces endomorphismes sont des automorphismes, et l'ensemble formé par eux est un p-groupe. En particulier, si G est un p-groupe fini avec G_k abelien, et a est un élément de G_{k-1}, l'application θ: $g\theta=g[g, a]^s$ est un automorphisme dont l'ordre est une puissance p^h de p; en outre, si p^s est l'exposant de G_k, et p^m est l'ordre de l'automorphisme intérieur induit dans G par a, on a $h \leq m+s-1$; et si c est divisible par p^t ($1 \leq s \leq t$), on a $h \leq s-t$. *G. Zappa* (Florence)

9538:

Itô, Noboru. Faktorisierung endlicher auflösbarer Gruppen. Math. Z. **74** (1960), 392–413.

Let G be a primitive soluble permutation group. By a classical theorem of Jordan, the degree of G is a prime power p^a and G contains a normal elementary abelian subgroup N of order p^a. The author considers the transitive abelian (hence regular) subgroups A of G. Theorem 1: If $p>3$, $A=N$. Theorem 2: If $p=3$, A is elementary abelian. $A=N$ when the Sylow 2-subgroups of G are abelian. Theorem 3: If $p=2$, the exponent of A is ≤ 4. $A=N$ when the order of G is not divisible by 3. (For earlier results of Ritt, Huppert and Itô, see Itô, Acta Math. Sci. Szeged **16** (1955), 207–228 [MR **17**, 579].) To prove these theorems, the author considers the following situation: H is a finite group with subgroups N, R, U, A such that: (a) N is a normal elementary abelian p-subgroup which is its own centralizer in H; (b) R is nilpotent of order prime to p and the centralizer of R in H is contained in NR; (c) U is an abelian p-subgroup which normalizes R; (d) $H=NRU$ and the order of H is the product of the order of N, R, U; (e) A is an abelian p-subgroup such that $H=ARU$ and $NA=AU=NU$. Under these hypotheses, he shows that, if $p>3$, $A=N$ and so on. The theorems follow by taking R as the radical

of M and U as $NA \cap M$, where M is the subgroup of G formed by the permutations leaving a given symbol fixed. The proofs are not easy and several cases have to be considered. {The proof of theorem 1 can be simplified by using theorem 2.1.1 of Hall and Higman, Proc. London Math. Soc. (3) **6** (1956), 1–42 [MR **17**, 344].}

G. E. Wall (Chicago, Ill.)

9539:

Feit, Walter. A theorem on factorizable groups. Proc. Amer. Math. Soc. **11** (1960), 658–659.

Soit G un groupe fini d'ordre impair, M un sous-groupe maximal $\neq 1$ de G, A un sous-groupe abélien de G qui a au moins un sous-groupe de Sylow cyclique. Alors, s'il est $G=AM$, G contient un sous-groupe normal $\neq 1$, qui est dans A ou dans M. La démonstration est basée sur un théorème de Wielandt [Math. Z. **40** (1935), 582–587]. Corollaires: (1) Soit G un groupe fini d'ordre impair, et soit $G=HA$, où H et A sont des sous-groupes propres de G, et A est abélien avec au moins un sous-groupe de Sylow cyclique. Alors G n'est pas simple; (2) Un groupe fini d'ordre impair avec un sous-groupe $\neq 1$ d'index premier n'est pas simple; (3) Un groupe fini G d'ordre impair, tel que $G=HA$, avec H résoluble et A cyclique, est résoluble.

G. Zappa (Florence)

9540:

Cohen, Eckford. Partitions in homogeneous, finite abelian groups. Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 1290–1291.

Some results concerning so-called extended even functions which will be proved in a later paper are here presented in group-theoretical terminology. See #9465.

H. Bergström (Göteborg)

9541:

Wendelin, Hermann. Einige hinreichende Bedingungen der Erweiterungstheorie. Arch. Math. **11** (1960), 91–94.

Let B be a commutative additive semigroup containing as a subsemigroup the additive semigroup N of natural integers. It is shown that if (i) the elements of N are cancellative elements of B (i.e., $a+x=a+y$ implies $x=y$ for a in N) and (ii) there exists a pair of elements a, b $(b>a)$ in N with $b+x=a$ for some x in B, then N is contained in a subgroup of B. Two similar results are proved.

G. B. Preston (Shrivenham)

9542:

Ševrin, L. N. On densely imbedded ideals of semigroups. Dokl. Akad. Nauk SSSR **131** (1960), 765–768 (Russian); translated as Soviet Math. Dokl. **1**, 348–351.

An ideal N of the semigroup T is dense in T [cf. E. S. Lyapin, same Dokl. **88** (1953), 13–15; errata, **92** (1953), 692; MR **15**, 395] if (i) every proper (i.e., not one-to-one) homomorphism of T induces a proper homomorphism of N, and (ii) for any ideal N of any semigroup S that properly contains T there exists a proper homomorphism of S which induces an isomorphism of N. This paper claims to prove the result: If N, a semigroup with zero, has a non-trivial annihilator, N cannot be a densely embedded ideal in any semigroup. The proof breaks down because it is assumed, in the notation of the paper, that when $K_{\alpha\beta}$ is non-empty then $K_{\alpha\beta}a_\gamma$ is a single element. The author's proof of this holds only if $a_\gamma \in aT$. The theorem holds however for commutative semigroups. It appears difficult to extend the author's method to any wider class of semigroups.

G. B. Preston (Shrivenham)

9543:

Munn, W. D. Semigroups satisfying minimal conditions. Proc. Glasgow Math. Assoc. **3**, 145–152 (1957).

A semigroup in which every set of principal left ideals has a minimal element with respect to inclusion is said to satisfy the condition M_l. The conditions M_r and M_f are defined analogously for right ideals and for two-sided ideals respectively. Let F be the set of generators of the principal ideal I and let J be its set of non-generators. The set F is called an $\mathfrak{f}$-class and F is said to be [completely] simple if either F or I/J is a [completely] simple semigroup. A semigroup is called semisimple if all of its $\mathfrak{f}$-classes are simple. These concepts were first introduced by J. A. Green [Ann. of Math. (2) **54** (1951), 163–172; MR **13**, 100].

The author begins by showing that a semigroup may satisfy any one of M_l, M_r and M_f without satisfying either of the other two, but that in a semisimple semigroup each of M_l and M_r implies M_f. Some related concepts are introduced. A semigroup satisfies $M_l^*[M_r^*]$ if, for each $\mathfrak{f}$-class F, the set of all principal left [right] ideals generated by elements of F has a minimal element. A semigroup is said to be completely semisimple if each $\mathfrak{f}$-class is completely simple. It is shown that a semisimple semigroup is completely semisimple if and only if it satisfies both M_l^* and M_r^*. In a completely semisimple semigroup the conditions M_l, M_r and M_f are equivalent. Semigroups which are unions of groups are characterized as semigroups S satisfying M_l^* and for which $a \in a^2S$ for all a in S [cf. R. Croisot, Ann. Sci. École Norm. Sup. (3) **70** (1953), 361–379; MR **15**, 680]. The paper ends with a discussion of various radicals in semigroups satisfying minimal conditions.

G. B. Preston (Shrivenham)

TOPOLOGICAL GROUPS AND LIE THEORY

See also 9873, 9874, 9875, 9876.

9544:

Rosenblatt, M. Limits of convolution sequences of measures on a compact topological semigroup. J. Math. Mech. **9** (1960), 293–305.

Es sei S eine kompakte Hausdorffsche topologische Semigruppe. Es bedeute $\mathfrak{B}$ den Körper aller Borelschen Teilmengen von S. Ist dann ν bzw. μ ein Mass auf $\mathfrak{B}$, so bedeutet $\nu \times \mu$ das Produktmass auf $\mathfrak{B} \times \mathfrak{B}$ und $\nu * \mu$ die Convolution von ν und μ, d.h.,

$$\nu * \mu(B) = \int C_B(xy)d(\nu \times \mu)((x, y))$$

für $B \in \mathfrak{B}$, wobei C_B die charackteristische Funktion von B und $C_B(xy)$ der Wert der Funktion C_B im Produkte xy der beiden Elemente $x \in S$, $y \in S$ ist. Bezeichnet man nun mit $\mathfrak{M}$ die Gesamtheit aller regulären Wahrscheinlichkeitsmasse μ auf $\mathfrak{B}$ und führt man eine Konvergenz in $\mathfrak{M}$ ein durch: $\mu_n \to \mu$ dann und nur dann, wenn $\int f d\mu_n \to \int f d\mu$ für

jede stetige Funktion f auf S gilt, und betrachtet man die Convolution als eine multiplikative Operation in $\mathfrak{M}$, so wird dann $\mathfrak{M}$ eine kompakte Hausdorffsche topologische Semigruppe. Für ein $\nu \in \mathfrak{M}$ wird die Folge $\nu^{(1)} = \nu$, $\nu^{(2)} = \nu^{(1)} * \nu$, $\cdots$, $\nu^{(n+1)} = \nu^{(n)} * \nu$, $\cdots$ als eine Convolutionsfolge erklärt. Es existiert dann

$$(1) \qquad \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \nu^{(j)} = \mu \in \mathfrak{M}$$

und es gilt $\mu * \nu = \nu * \mu = \mu^{(2)} = \mu$. Bezeichnet man mit $\Sigma(\nu)$ das Spektrum eines Masses $\nu \in \mathfrak{M}$, so bildet die abgeschlossene Hülle der Vereinigung $\bigcup_n \Sigma(\nu^{(n)})$ eine Untersemigruppe von S. Fällt diese Untersemigruppe mit S zusammen, so sagt man das Spektrum $\Sigma(\nu)$ des Masses $\nu \in \mathfrak{M}$ erzeugt die Semigruppe S. In diesem Falle fällt das Spektrum des Limesmasses (1) mit dem Kern K der Semigruppe, d.h., mit dem Durchschnitt $\bigcap_e SeS$ für alle idempotente Elemente e von S, zusammen. Insbesondere, falls $kS = k$ $[Sk = k]$ für jedes $k \in K$ gilt, dann existiert auch $\lim_{n\to\infty} \nu^{(n)}$ und ist gleich dem Limesmass μ von (1), also hat auch K als Spektrum. Letzterer Fall ist in manchen Strukturproblemen für stationäre Prozesse von Interesse.

D. A. Kappos (Athens)

9545:

Hahn, F. J. Some embeddings, recurrence properties, and the Birkhoff-Markov theorem for transformation groups. Duke Math. J. **27** (1960), 513–525.

Let (X, T) be a transformation group whose phase space X is a compact Hausdorff space and whose phase group T is locally compact. Relative to a direct class S of subsets of T with positive finite Haar measure, the author generalizes the classical notion of a center of attraction of a dynamical system to (X, T) as a closed invariant subset Q of X such that the limiting relative sojourn of each point of X in each neighborhood of Q is 1. In this context he elegantly establishes various properties of centers of attraction, including the existence of a least. The (topological) center of (X, T) is defined in a purely topological way as the greatest closed invariant subset C of X such that the transformation group (C, T) is regionally recurrent. A desired theorem ("Birkhoff-Markov") for (X, T) is that the center is a center of attraction. To attain this theorem the class S, relative to which limits are taken, must presumably be specialized. So far, the theorem is known only for generative groups T, that is, abelian groups generated by some compact neighborhood of the identity. By a structure theorem of A. Weil, generative groups T are exactly those of the form $K \times I^n \times R^m$, where K is a compact abelian group, I is the discrete additive group of integers, R is the additive group of reals with its usual topology, and n and m are non-negative integers. The author first proves the Birkhoff-Markov theorem for $T = R^m$, then develops an embedding process for general transformation groups permitting the embedding of I^n into R^n with the phase space enlarged, shows the compact factor K has no effect on the notions involved, and thence derives the desired Birkhoff-Markov theorem for generative groups $T = K \times I^n \times R^m$. His specialized class S consists essentially of expanding hypercubes centered at the origin with edges parallel to the coordinate axes.

By substantially different methods, Bernard [Duke Math. J. **18** (1951), 307–319; MR **12**, 720] arrives at similar results for generative T by taking S to be the powers of a translate of a neighborhood of the identity with compact closure.

W. G. Gottschalk (New Haven, Conn.)

9546:

Satake, Ichiro. On representations and compactifications of symmetric Riemannian spaces. Ann. of Math. (2) **71** (1960), 77–110.

Let $G = G/K$ be a symmetric Riemannian space, where G is a semi-simple Lie group and K a maximal compact subgroup of G. This paper is concerned with the compactification of S. (By a compactification of S, we mean a compact Hausdorff space $\bar{S}$ containing S as a dense subset such that the action of G on S can be extended continuously to $\bar{S}$.) Let $\mathfrak{g}$ be the Lie algebra of G, and assume G to act almost effectively on S. Then to each faithful irreducible representation ρ of $\mathfrak{g}$ over an n-dimensional complex vector space, there corresponds an isometric imbedding of S into the space $\mathscr{P}_n^1$ of all positive definite hermitian matrices of degree n and determinant 1 with the property that each symmetry of S can be extended to a symmetry of $\mathscr{P}_n^1$. Denote by $P(\mathscr{H}_n)$ the real projective space associated with the real vector space $\mathscr{H}_n$ of all hermitian matrices of degree n. In the natural manner, $\mathscr{P}_n^1$ can be regarded as a subset of $P(\mathscr{H}_n)$, and thus an imbedding $S \to P(\mathscr{H}_n)$ is obtained. This imbedding will be denoted by the same letter ρ as the representation of $\mathfrak{g}$. The closure $\overline{\rho(S)}$ of $\rho(S)$ in $P(\mathscr{H}_n)$ is then a compactification of S.

The author gives a thorough study of the structure of $\overline{\rho(S)}$. In fact, let $\mathfrak{k}$ be the Lie algebra of K, $\mathfrak{p}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ (with respect to the Killing form), and $\mathfrak{h}^-$ a maximal abelian subalgebra in $\mathfrak{p}$. By using the system of roots of $\mathfrak{g}$, the author constructs a finite subset Δ^- of $\mathfrak{h}^-$. This subset is not unique. Nevertheless, any two such Δ^- are conjugate under the adjoint group of $\mathfrak{g}$. Let μ_ρ be the restriction, on $\mathfrak{h}^-$, of the dominant weight (with respect to a Cartan subalgebra of $\mathfrak{g}$ which contains $\mathfrak{h}^-$) of the representation ρ, and let Δ_ρ^- denote the totality of elements in Δ^- not orthogonal to μ_ρ. Then Δ_ρ^- intersects each connected component of Δ^-. (A subset of Δ^- is called connected if it cannot be written as the union of two non-empty mutually orthogonal subsets.) The author proves that $\overline{\rho(S)}$ is completely determined by Δ_ρ^- and does not depend on ρ essentially. Moreover, there exists a one-to-one correspondence between these compactifications and the subsets of Δ^- which intersect every connected component of Δ^-.

When S is a symmetric bounded domain of the classical type, one gets a compactification by taking the closure of the classical model of S in the complex euclidean space. This compactification is a particular case of those discussed in this paper. In fact, it corresponds to the case by which Δ_ρ^- consists of the restrictions on $\mathfrak{h}^-$ of the non-compact roots in the sense of Harish-Chandra [Amer. J. Math. **77** (1955), 743–777; MR **17**, 282].

H.-C. Wang (Evanston, Ill.)

9547:

Curtis, Charles W. On projective representations of certain finite groups. Proc. Amer. Math. Soc. **11** (1960), 852–860.

L'auteur a construit dans un travail antérieur [J. Math. Mech. **9** (1960), 307–326; MR **22** #1634] des représentations projectives irréductibles du groupe G des

automorphismes d'une algèbre de Lie "de type classique" sur un corps algébriquement clos Ω de caractéristique $p>0$; soit G_0 le sous-groupe de G engendré par les sous-groupes à un paramètre $\xi\to\exp(\mathrm{ad}(\xi e_\alpha))$, où ξ varie dans le corps premier Ω_0 de Ω, et pour chaque racine α (relative à une algèbre de Cartan fixée) e_α est une base du sous-espace correspondant dans l'algèbre de Lie considérée. L'auteur prouve alors que toute représentation irréductible de G reste irréductible par restriction à G_0 et que deux représentations irréductibles non équivalentes de G restent non équivalentes par restriction à G_0. Les méthodes sont du type habituel en théorie des algèbres de Lie (considération des poids dominants). *J. Dieudonné* (Paris)

MISCELLANEOUS TOPOLOGICAL ALGEBRA

See also 9511, 9520.

9548:

Storey, Charles R. The structure of threads. Pacific J. Math. **10** (1960), 1429–1445.

A thread is a connected topological semigroup in which the topology is the interval topology induced by a total order. The structure of positive threads and standard threads was given by P. S. Mostert and A. L. Shields in the separable case and in the general case by Clifford [for the terminology and historical development, see A. H. Clifford, Bull. Amer. Math. Soc. **64** (1958), 305–316; MR **20** #7070]. The following is shown. (i) If S is a thread with a zero as a least element and if $S^2=S$, then S is a standard thread, or S is a standard thread with its identity removed, or S is a positive thread. (ii) Let S be a thread with proper minimal ideal K, and let $S^2=S$. Let $R=[t|k\leq t$ for each k in $K]$, $L=[t|t\leq k$ for each k in $K]$. Then, passing to the order dual if necessary, $R^2=R$ and is thus completely described by (i). Moreover, if $L\subset L^2$ then $L=L^2$ as well. Finally, K does not separate R^2, L^2, LR or RL, and the multiplication in S/K is monotone with respect to $<$. (iii) If e and f are any two idempotents in a thread, then the closed interval between them is a subthread.

T. S. Wu (New Orleans, La.)

9549:

Warner, Seth. Compact rings and Stone-Čech compactifications. Arch. Math. **11** (1960), 327–332.

Some theorems concerning compact topological rings are proved. Among them are the following. (1) The ring of all continuous functions with compact support from a locally compact, totally disconnected space Y into a finite field admits a compact topology that is compatible with its ring structure if and only if Y is the Stone-Čech compactification of a discrete subspace. (2) A semisimple ring admits at most one compact topology compatible with its ring structure. *M. Jerison* (Lafayette, Ind.)

FUNCTIONS OF REAL VARIABLES

See also 9506, 9565.

9550:

Boas, Ralph P., Jr. ★A primer of real functions. The Carus Mathematical Monographs, No. 13. Published by The Mathematical Association of America; distributed by John Wiley and Sons, Inc.; New York, 1960. xiii+189 pp. $4.00.

The author states his purpose in the following words: "My idea is to go reasonably far in a few directions with a minimum amount of special terminology." One finds that the special symbolism is restricted to those for inclusion, union and intersection $\subset$, $\cup$ and $\cap$. With this meagre equipment the author proves such theorems as the following. A non-empty closed set E in a complete metric space is also a complete metric space. A complete metric space is of the second category. The elements of the space C of continuous functions that have, even at one point, a finite derivative, even on one side, form a set of the first category in C. A continuous function is uniformly continuous on any compact set in its domain. Baire's theorem that a complete metric space is of the second category.

There is a fairly complete coverage of derivatives, an example of a continuous curve that fills a square, and a proof of the theorem that there cannot be a continuous curve that passes through each point of a square precisely once.

Exterior measure of a set E is defined and it is proved that at almost all of E the density, in terms of exterior measure, is unity.

This summary of the contents is not exhaustive. It is given to indicate that the author has gone "reasonably far" in a few directions. There is, in addition, a description of Cantor's approach to the real number system, a study of infinite series including the ideas of convergence and summability, the properties of continuous functions and sequences of functions, a description of Baire's classifications of functions, and the main ideas and results of uniform convergence. Ordinary integration is used to show, among other things, that a uniformly convergent series of functions is integrable term by term, but there is no development of the theory of integration as such.

The text is a valuable contribution to mathematical literature in that it sets forth in simple language and in short space the parts of the real variable theory that are essential to further study in the various fields of mathematics. *R. L. Jeffery* (Wolfville, N.S.)

9551a:

Favard, J. ★Cours d'analyse de l'École Polytechnique. Tome I: Introduction. Opérations. Cahiers Scientifiques, Fasc. 26. Gauthier-Villars, Paris, 1960. 675 pp. 90 NF; $18.50.

9551b:

Favard, J. ★Cours d'analyse de l'École Polytechnique. Tome II: Représentations. Fonctions analytiques. Cahiers Scientifiques, Fasc. 26. Gauthier-Villars, Paris, 1960. v+578 pp. 80 NF; $17.00.

These are the first two volumes of a three-volume work intended for students at l'École Polytechnique. The treatise contains a mine of information, presented carefully and clearly. More than one-fourth of it is in the form of nonroutine exercises or supplementary discussion. The Introduction consists of 245 pages of background material (see below). Part I, Operations (218 pp.), deals with differential and integral calculus, including vector analysis,

Riesz-Fischer theorem, Fubini's theorem. Part II, representations (178 pp.), includes the Weierstrass approximation theorem, Fourier series, elements of the theory of distributions. Part III, Analytic functions (390 pp.), includes Picard's theorem, elliptic functions, several variables, algebraic functions. (To appear—Vol. III: Part IV, Theory of equations.) Random selections from section headings: implicit function theorems, Lagrange multipliers, Bernoulli and Legendre polynomials, completely additive set functions, gamma function, linear representations in L^2, Féjer sums, Bernstein polynomials, Lobatchefskian geometry, Bloch's theorem, normal families, Laplace transform, Weyl's theorem, Riemann-Roch theorem, theta functions.

The author assumes that his students are already familiar with the real number system; accordingly, the Introduction is restricted to the following preliminary material: set theory, algebra, topology, function spaces, infinite series and products, functions of bounded variation. Topics mentioned in these chapters and their appendices include equivalence and order relations, cardinals, Bernstein equivalence theorem, Liouville's transcendental numbers, proofs of transcendence of e and π, the denumerable ordinals, group, ring, field, ideal, vector space, matrices (characteristic polynomial, unitary matrices, elementary divisors, normal forms), euclidean spaces, metric spaces, normed vector spaces, continuity, convergence, Cauchy sequences and completeness, compactness, Borel-Lebesgue lemma, uniform continuity, Bolzano-Weierstrass lemma, connectedness, Jordan curves, simplexes, complexes, orientability, homology, homotopy, p-adic numbers, Cantor-Bendixson theorem, Urysohn's lemma, one-point compactification, Peano curves, fixed-point theorem, Banach algebras, Dini's lemma, Ascoli's theorem, semi-continuity, Banach-Steinhaus theorem, Abel's lemma, summability methods, weighted means. The reader is now ready to tackle the derivative.

L. Gillman (Rochester, N.Y.)

9552:

Nečas, Jindřich. Über Grenzwerte von Funktionen, welche ein endliches Dirichlet'sches Integral haben. Apl. Mat. **5** (1960), 202–209. (Czech and Russian summaries)

Gegeben ist eine rektifizierbare Jordankurve Γ. Die Frage ist, welche stetigen Funktionen auf Γ sich in das Innere von Γ als Funktionen der Klasse C^2 mit endlichem Dirichletintegral fortsetzen lassen. Unter gewissen Bedingungen lässt sie sich auf den einfacheren Fall zurückführen, wo Γ die Kreisperipherie ist; z.B., wenn Γ eine stückweise stetig variierende Tangente besitzt. Dem verschiedentlich behandelten Fall der Kreislinie kann Verf. folgendes beifügen: Ist f absolut stetig und gehört f' zur Klasse L^p, $p>1$, so ist die obige Ausdehnung möglich. Erwähnt sei noch dieses Resultat: Gilt $\int_0^{2\pi}|f(s+h)-f(s)|^2ds \leq M|h|$, so gehört f zur Klasse L^p, $p>1$.

A. Pfluger (Zürich)

9553:

Sholander, Marlow. Analytical expressions and elementary functions. Amer. Math. Monthly **67** (1960), 213–214.

Informal discussion of functions generated by "formulas".

9554:

Kiesewetter, H. Eine Bemerkung über partielle Differentiationen bei N. H. Abel. Publ. Math. Debrecen **5** (1958), 265–268.

If $\alpha(x, y)$ has the derivatives α_x and α_y, we can define an associated differential operator by $D_\alpha = \alpha_y(\partial/\partial x) - \alpha_x(\partial/\partial y)$. Then D_α obeys sum, product and chain rules like those of elementary calculus, and special rules hinging upon the relation of D_α to a directional derivative along the curves $\alpha(x, y) = \text{const}$. Writing D_α as a determinant and generalizing to n variables allows a neat formulation of certain matters of functional dependence.

R. M. Redheffer (Los Angeles, Calif.)

9555:

Montel, Paul. Sur les propriétés périodiques des fonctions. C. R. Acad. Sci. Paris **251** (1960), 2111–2112.

A continuous function which has two irrationally related periods must be constant. The author observes that the hypothesis can be weakened to the assumption that $f(x+w)-f(x) \geq 0$ and $f(x+w')-f(x) \leq 0$ where w'/w is positive and irrational. An analogous result characterizes linear functions by means of pairs of inequalities using second differences $f(x+2w)-2f(x+w)+f(x)$.

R. C. Buck (Madison, Wis.)

9556:

Sibagaki, Wasao. Reorganizing calculus up-to-date. Japan. J. Math. **29** (1959), 130–137.

The author attempts to unify the traditional (pre-Courant) approach to calculus with the point of view expressed by Menger and with the more modern treatments of analysis. Difficulties, both conceptual and practical, arise in the treatment of functions of several variables. In particular, the author tries to retain the notations dy, dy/dx and df/dx, along with dh/dg and $D^{-1}f$.

R. C. Buck (Madison, Wis.)

9557:

Marcus, S. Sur une propriété descriptive analogue à la propriété N de Lusin. Colloq. Math. **7** (1959/60), 213–220.

The paper is concerned with properties of continuous real functions defined on the unit interval and having the property N in the sense of category, i.e., transforming every set of the first category into a set of the first category. In particular, it is proved that for any such function f each interval contains a subinterval on which f is monotone. Moreover, the set of all x's for which the inverse image $f^{-1}(x)$ is infinite is of the first category. It is also proved, by a counterexample, that the converse implication is not true.

K. Urbanik (Wrocław)

9558:

Goffman, Casper. Functions of finite Baire type. Amer. Math. Monthly **67** (1960), 164–165.

From the author's summary: "In this note, the [function space] method is applied to the functions of finite Baire type to show that (a) there are sequences of functions of finite Baire type which converge uniformly to functions not of finite Baire type, but that (b) there are functions of Baire type ω_0 which are not the limits of any uniformly convergent sequences of finite Baire type."

T. A. Botts (Charlottesville, Va.)

9559:

Elyash, Ernest S.; Laush, George; Levine, Norman. On the product of two uniformly continuous functions on the line. Amer. Math. Monthly **67** (1960), 265–267.

Where K is the class of non-negative uniformly continuous functions on the interval $[1, \infty]$, let K^* be the class of all functions $f(x)$ in K such that $f(x)g(x)$ belongs to K whenever $g(x)$ does. Typical result: $f(x)$ belongs to K^* if and only if $xf(x)$ belongs to K.

T. A. Botts (Charlottesville, Va.)

9560:

Džafarov, A. S. Some properties of functions of several variables. Issledovaniya po sovremennym problemam teorii funkiĭ kompleksnogo peremennogo, pp. 537–544. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960. (Russian)

Generalizations of the embedding theorems of Nikol'skiĭ [Mat. Sb. (N.S.) **33** (**75**) (1953), 261–326; MR **16**, 453] and of compactness theorems of Kudryavcev [Uspehi Mat. Nauk **9** (1954), no. 1 (59), 111–120; MR **16**, 453] to classes of functions of several variables somewhat more general than the classes $H_p^{(r_1,\cdots,r_n)}$ of Nikol'skiĭ.

G. G. Lorentz (Syracuse, N.Y.)

9561:

Hsiang, Fu Cheng. On differentiable functions. Bull. Amer. Math. Soc. **66** (1960), 382–383.

Let f be a real function differentiable on the interval $[a, b]$. For $\alpha<\beta$ put $E_{\alpha\beta}=\{x|a\leq x\leq b,\ \alpha<f'(x)<\beta\}$. The author proves the following theorem: There is a set $\{I_i\}$ of nonoverlapping and nonabutting open sub-intervals of $[a, b]$ such that $E_{\alpha\beta}$ is void in each interval I_i and metrically dense everywhere in $[a, b]-\bigcup I_i$.

N. Dinculeanu (Bucharest)

9562:

Goffman, Casper; Neugebauer, C. J. On approximate derivatives. Proc. Amer. Math. Soc. **11** (1960), 962–966.

This note contains brief proofs of the following results (due essentially to Khintchine and Tolstoff). Assume the real-valued function f has a finite approximate derivative f_{ap}' everywhere on the real interval I_0. Then: (1) f_{ap}' is of Baire class 1; (2) f_{ap}' maps intervals into intervals; (3) the mean value theorem for f holds in terms of f_{ap}'; (4) the ordinary derivative of f exists except on a nowhere dense set.

L. M. Graves (Chicago, Ill.)

9563:

Vaĭnšteĭn, I. A.; Kreĭnes, M. A. Sequences of functions of the form $f(X(x)+Y(y))$. Uspehi Mat. Nauk **15** (1960), no. 4 (94), 123–128. (Russian)

The subject is functions φ on $[0, 1]\times[0, 1]$ which can (for some continuous f, X, Y) be represented as $\varphi(x, y)=f(X(x)+Y(y))$ there. In particular, for φ to be of this sort, is it sufficient that it be the uniform limit of functions which are? V. I. Arnol'd [same Uspehi **12** (1957), no. 2 (74), 119–121; MR **19**, 841] showed it is not sufficient (one counterexample being $\varphi(x, y)=xy$). Now the authors show that it is sufficient if the further hypothesis is adjoined that φ is strictly monotonic in each variable separately.

C. Davis (Providence, R.I.)

9564:

Callender, E. David. Hölder continuity of n-dimensional quasi-conformal mappings. Pacific J. Math. **10** (1960), 499–515.

Let $x\to u=f(x)$, where $x=(x_1, \cdots, x_n)$ and $u=(u_1, \cdots, u_n)$ define a map from E_n to E_n. The author calls the map quasi-conformal if it is of class C^1, the Jacobian $J=\det(\partial f/\partial x)\geq 0$ and $|\nabla f|^2\leq nKJ^{2/n}$, where $|\nabla f|^2=\sum\sum(\partial u_i/\partial x_j)^2$ and $K\geq 1$ is a constant. It is called near quasi-conformal if $J\geq 0$ and $|\nabla f|^n\leq(nK)^{n/2}J+K_1$, $K\geq 1$, $K_1>0$. The main result is that if $x\to f$ is near quasi-conformal on an open set A, then f is uniformly Hölder continuous on every compact subset B of A with a Hölder exponent $\mu=\mu(n, K)$ and a Hölder constant $C=C(n, K, d)$, $d=\operatorname{dist}(B, \text{bdry } A)$. The proof is a (non-straightforward) analogue of arguments due to Morrey for $n=2$, $K_1=0$ [Trans. Amer. Math. Soc. **43** (1938), 126–166] and Nirenberg for $n=2$ [Comm. Pure Appl. Math. **6** (1953), 103–156; MR **16**, 367]. It depends on estimating the growth of $D(r)=\int|\nabla f|^n dx$ over spheres $|x-x_0|\leq r$ and showing that $D(r)\leq Lr^{n\mu}$, $0<\mu<1$, implies Hölder continuity of f. The results are used to obtain theorems about removable singularities of f and about one-to-one quasi-conformal mappings of spheres onto spheres.

P. Hartman (Baltimore, Md.)

MEASURE AND INTEGRATION

See also 9936, B9989.

9565:

Колмогоров, А. Н. [Kolmogorov, A. N.]; Фомин, С. В. [Fomin, S.V.]. ★Элементы теории функций и функционального анализа. II: Мера, интеграл Лебега, гильбертого пространство [Elements of the theory of functions and of functional analysis. II: Measure, Lebesgue integral, Hilbert space]. Izdat. Moskov. Univ., Moscow, 1960. 119 pp. 3.50 r.

This fascicule is a continuation of the first of the same title [1954; translation, Graylock Press, Rochester, N.Y., 1957; MR **16**, 1122; **19**, 44]. The table of contents is as follows. Chapter V: Measure theory. Chapter VI: Measurable functions. Chapter VII: The Lebesgue integral. Chapter VIII: Functions with integrable square. Chapter IX: Abstract Hilbert space, integral equations with symmetric kernel. The material is presented with great care, and is accompanied by a number of interesting comments and remarks, which are in harmony with the broad viewpoint possessed by the authors. The book is useful as a short introduction to the theory of measure and of Hilbert space, although it is far from being a complete textbook for the serious student.

E. Hewitt (Seattle, Wash.)

9566a:

Kolmogorov, A. N.; Fomin, S. V. ★Elements of the theory of functions and functional analysis. Vol. 2: Measure. The Lebesgue integral. Hilbert space. Translated from the first (1960) Russian ed. by Hyman Kamel and Horace Komm. Graylock Press, Albany, N.Y., 1961. ix+128 pp. $4.00.

9566b:

Kolmogorov, A. N.; Fomin, S. V. ★**Measure, Lebesgue integrals, and Hilbert space.** Translated by Natascha Artin Brunswick and Alan Jeffrey. Academic Press, New York-London, 1961. xii+147 pp. $4.00.

Two translations of the Russian text reviewed above. The Graylock edition contains also numerous exercises added by Kamel.

9567:

Dinges, Hermann. Über das simultane Teilen von Mengen mittels Linearscharen messbarer Funktionen. Bayer. Akad. Wiss. Math.-Nat. Kl. S.-B. **1959**, 415–429 (1960).

Veranlasst durch die Betrachtungen von H. Steinhaus [Fund. Math. **33** (1945), 245–263; MR 8, 164] betrachtet Verf. sehr allgemeine Teilen (insbesondere Halbirungen) von Mengen mittels gegebener Funktionen und Mengen. Als Hilfsmittel werden topologische und Masstheoretische Ideen angewandt. Sei $\mathfrak{M}$ ein σ-Körper von Teilmengen einer Grundmenge Ω, und $\mu|\mathfrak{M}$ ein Mass; sei n eine natürliche Zahl und $A_1, \cdots, A_n \in \mathfrak{M}$ mit $\mu A_\nu > 0$; für jedes $M \in \mathfrak{M}$ werden die n Funktionen

$$\tilde{m}_\nu M = 2\frac{\mu(M \cap A_\nu)}{\mu A_\nu} - 1$$

und das "Mass-n-tupel" $\tilde{m} = [\tilde{m}_1, \cdots, \tilde{m}_n]$ definiert; für jede messbare Funktion $f|\Omega$ wird das Mass-n-tupel mf als $\tilde{m}\{x; fx < 0\}$ erklärt. Seien $f_0, \cdots, f_n$ $n+1$ messbare Funktionen auf Ω, die auf jeder Teilmenge $M \subset \bigcup A_\nu$ mit $\mu M > 0$ linear unabhängig sind; d.h., $\tilde{m}\{x; \sum \alpha_\nu f_\nu(x) = 0\} = [-1, -1, \cdots, -1]$ für jede nichttriviale Folge der reellen Zahlen $\alpha_0, \alpha_1, \cdots, \alpha_n$. Es werden die linearen Kombinationen $g(x)$ der $f_0, \cdots, f_n$ betrachtet; insbesondere die grössten Klassen derjenigen gx die, sich nur um einen Faktor >0 unterscheiden, werden als Schnitte durch Ω genannt; alle Schnitte werden durch

$$\rho\Big(\sum_{\nu=0}^{n} a_\nu^{(1)} f_\nu, \sum_{\nu=0}^{n} a_\nu^{(2)} f_\nu\Big) = \arccos \frac{\sum a_\nu^{(1)} a_\nu^{(2)}}{(\sum a_\nu^{(1)2} \sum a_\nu^{(2)2})^{1/2}}$$

metrisiert. Haupttheorem: Für jedes $m^* = [m_1^*, \cdots, m_n^*]$ mit $|m_\nu^*| < 1$ zerfällt die Menge der Schnitte P durch Ω mit $m(P) = m^*$ in genau zwei zusammenhängende Komponenten N_1, N_2 mit der Eigenschaft: Führt man ein $P_1 \in N_1$ stetig über in ein $P_2 \in N_2$, so nimmt auf dem Wege mindestens ein m_ν einen der Werte ± 1 an.

Unter den angeführten Voraussetzungen über A_ν and f_ν, existiert eine lineare Kombination $g(x)$ der f_ν, so dass $m(g) = [0, 0, \cdots, 0]$ ist, d.h., der durch $\{x; gx < 0\}$ und $\{x; gx \geq 0\}$ definierte Schnitt alle Mengen $A_1, \cdots, A_n$ halbiert. *Đ. Kurepa* (Zagreb)

9568:

Nikodým, Otton-Martin. Sur la mesure non archimédienne effective sur une tribu de Boole arbitraire. C. R. Acad. Sci. Paris **251** (1960), 2113–2115.

9569:

Korovkin, P. P. Set capacity and singular functions. Dokl. Akad. Nauk SSSR **129** (1959), 250–253. (Russian)

Let ϕ be a non-decreasing singular function on $[-1, 1]$. Define $\beta_0 = -1 < \alpha_1 < \beta_1 < \cdots < \beta_n < \alpha_{n+1} = 1$, $A_n = \bigcup_1^n [\alpha_k, \beta_k]$, $B_n = \bigcup_0^n [\beta_k, \alpha_{k+1}]$. If $0 < q < 2$, let $\Delta_n(q)$ be the infimum of $\int_{B_n} d\phi(x)$ for all A_n of Lebesgue measure $q < 2$. Then $\Delta_n(q) \to 0$ if $n \to \infty$. If the infimum is taken for all A_n of transfinite diameter q, however, a function ϕ is exhibited for which the infimum is bounded from below by a strictly positive constant for every q with $0 < q < 1/2$. *J. L. Doob* (Urbana, Ill.)

9570:

Comfort, W. W.; Gordon, Hugh. Vitali's theorem for invariant measures. Trans. Amer. Math. Soc. **99** (1961), 83–90.

Les auteurs donnent un théorème du type de Vitali, avec des hypothèses d'une nature nouvelle sur les recouvrements fermés considérés. Les hypothèses usuelles introduisent en chaque point x de l'espace topologique mesuré que l'on considère une famille dénombrable de "boules" S_n dont la mesure tend vers 0 avec $1/n$; et on postule l'existence d'un $\alpha > 0$ (dépendant de x) et d'un système (U_n) d'ensembles du recouvrement, contenant x, et tels que $U_n \subset S_n$ et $\mu(U_n) > \alpha \cdot \mu(S_n)$ pour tout n. Ici les hypothèses des auteurs font intervenir des conditions d'uniformité sur le recouvrement; ces conditions d'uniformité sont introduites à l'aide d'un groupe transitif de transformations de l'espace mesuré (conservant la mesure), mais il semble au rapporteur qu'on pourrait les formules à l'aide du langage des structures uniformes, sans faire intervenir de groupe. On a donc ici une famille de "boules" S_n relatives à un point fixe x_0 de l'espace, de mesure extérieure tendant vers 0, avec $\mu^* S_{n+1} \geq \varepsilon \mu^* S_n$ pour un $\varepsilon > 0$ fixe, et une condition qui affirme grosso modo que pour les opérations du groupe, S_{n+1} est plus petit que "la moitié" de S_n. On impose alors aux ensembles du recouvrement la condition que pour tout x, tout voisinage V de x il y a un $n > 0$ et un ensemble du recouvrement U tel que par une translation convenable, U soit contenu dans un S_n, et $\mu(U) > \alpha(x) \cdot \mu^*(S_n)$. Moyennant cette condition, la conclusion du théorème de Vitali usuel est valable. Les auteurs remarquent que leurs conditions peuvent être remplies même lorsque les conditions usuelles de "régularité" ne le sont pas (ces dernières impliquant en chaque point x une uniformité en n, alors qu'ici l'uniformité est en x); les auteurs montrent aussi comment, sur un groupe de Lie, on peut définir des systèmes de "boules" S_n ayant les propriétés voulues. *J. Dieudonné* (Paris)

9571:

Nachbin, Leopoldo. ★**Integral de Haar.** Textos de Matemática, No. 7. Instituto de Física e Matemática, Universidade do Recife, 1960. (Portuguese) 238 pp.

This leisurely exposition of the theory of Haar measure is intended for students with a minimum amount of knowledge of anything beyond calculus (essentially, the first definitions in general topology and the theory of Banach spaces); everything else which is needed (starting, for instance, with compact spaces, topological groups, and semi-continuous functions) is treated in the book itself. This includes in particular the part of integration theory on locally compact spaces which is required for the main results on Haar measure; this turns out to be much less than one would expect, and does not even include Lebesgue's convergence theorem or the general form of

the Lebesgue-Fubini theorem; it is developed by the author after the method of Bourbaki. This preliminary material occupies about one half of the book; the rest is concerned with the basic properties of invariant and relatively invariant measures on locally compact groups and their homogeneous spaces; quasi-invariant measures (on homogeneous spaces for which no relatively invariant measure exists) are not considered. The book is well and clearly written, and one of its attractive features is the large number of examples treated by the author as well as many remarks which aim at locating the theory both historically and in relation with neighboring parts of mathematics. It certainly constitutes one of the best introductions to the subject. *J. Dieudonné* (Paris)

9572:

Blum, J. R.; Hanson, D. L. On the mean ergodic theorem for subsequences. Bull. Amer. Math. Soc. **66** (1960), 308–311.

The authors prove the following theorem: If (Ω, A, P) is a probability space, and if T is a 1–1 bimeasurable mapping of Ω onto Ω, then T is a strongly mixing transformation in Ω if and only if for every choice of $0 \leq p < \infty$ and every strictly increasing sequence $\{k_i\}$ of positive integers and every $f \in L_p(\Omega)$,

$$n^{-1} \sum_{i=1}^{n} f(T^{k_i}x) \to \int_\Omega f\,dP$$

in the norm of $L_p(\Omega)$. *A. Beck* (Madison, Wis.)

9573:

Sirokov, F. V. Solution of a problem of Halmos (a concept of recurrence for numerical sequences). Uspehi Mat. Nauk **15** (1960), no. 4 (94), 185–192. (Russian)

P. R. Halmos [*Lectures on ergodic theory*, Publ. Math. Soc. Japan, No. 3, Math. Soc. Japan, 1956; MR **20** #3958; p. 96] posed the question whether a concept of "recurrence" for real sequences can be defined so that: (1) if $\{a_i\}$ is a recurrent real sequence, then $n^{-1}\sum_{i=0}^{n-1} a_i$ converges to a finite limit; (2) if X is a (totally sigma-finite) measure space, if T is a measure-preserving transformation of X into X, and if f is an integrable real function on X, then $\{f(T^i x)\}$ is recurrent almost everywhere on X.

The author answers the question affirmatively by defining a real sequence $\{a_i\}$ to be recurrent provided that: (a) $\lim_{n\to\infty} n^{-1}$ card$\{i \mid 0 \leq i \leq n-1$ and $a_i \leq t\}$ exists for t belonging to a dense subset of the real line; (b) for each $\varepsilon > 0$ there exists $A > 0$ such that $|n^{-1}\sum_{i=0}^{n-1} a_i^+(A)| < \varepsilon$ and $|n^{-1}\sum_{i=0}^{n-1} a_i^-(-A)| < \varepsilon$ for all n, where $a_i^+(A) = 0$ if $a_i \leq A$, $a_i^+(A) = a_i$ if $a_i > A$, $a_i^-(-A) = a_i$ if $a_i \leq -A$, $a_i^-(-A) = 0$ if $a_i > -A$. In his proof the author makes use of the ergodic theorem.

{The reviewer wonders whether this notion of "recurrence" has any connection with the notion of recurrence in topological dynamics.}

W. H. Gottschalk (New Haven, Conn.)

9574:

Rishel, Raymond. Area as the integral of lengths of contours. Trans. Amer. Math. Soc. **97** (1960), 95–119.

Let Q be the unit square in E^2, $T: Q \to E^2$ be continuous and $A(T)$ be the Lebesgue area of T. Let $T = l \circ m$ be a monotone-light factorization of T and $M = m(Q)$. If $p, q \in Q$ then $G(p, q, T) = \inf$ length $l \circ g$ where g is continuous on $[0, 1]$ into M, $g(0) = m(p)$, and $g(1) = m(q)$. Let $D(p, t) = \{q \in Q \mid G(p, q, T) > t\}$ and $L(p, t)$ be the Cesari length of T restricted to the boundary of $D(p, t)$. Theorem: If T is non-degenerate then $A(t) \geq \int_0^\infty L(p, t)\,dt$ for all $p \in Q$. If $A(t) < \infty$ there exists p such that the equality holds. Some tools are the result (independently obtained) of Federer and Young on partial area, Morrey's representation theorem, cyclic additivity and a modification of Morse's μ-length due to the reviewer.

E. Silverman (Lafayette, Ind.)

FUNCTIONS OF COMPLEX VARIABLES

See also 9712, 9872.

9575:

Romanovskii, P. I. ★Mathematical methods for engineers and technologists. Translated from the Russian by M. Slater. Edited by T. Kóváry. Pergamon Press, New York-Oxford-London-Paris, 1961. xiv + 253 pp. $8.50.

The first chapter, on Fourier series and integrals, establishes representation theorems and presents the various forms of the series and integrals with some illustrative examples. Orthogonal polynomials are discussed briefly. This is followed by a chapter on elements of vector analysis. The third and longest chapter presents the classical theory of functions of a complex variable, including theory of residues and conformal mapping, but not applications of mapping. A chapter entitled "Some special functions" introduces the gamma, Bessel, sine-integral and related functions. The final chapter presents the basic properties of the Laplace transformation, with some applications to ordinary differential equations and difference equations. Throughout the book the exposition is sound and concise. The lithoprint copy is fairly easy to read. The book contains no index, no exercises for the reader, and no references except for two cited in the preface. *R. V. Churchill* (Ann Arbor, Mich.)

9576:

Dolženko, E. P. On the differentiation of complex functions. Dokl. Akad. Nauk SSSR **130** (1960), 17–20 (Russian); translated as Soviet Math. Dokl. **1**, 8–11.

Let E be a set in the extended complex plane and let $f(z)$ be an extended complex-valued function defined on E. A number a is said to be a derivative number of $f(z)$ at $\zeta \in E$ (taken with respect to E) if there exists a sequence of points $\{z_n\}$ in E which converge to ζ, such that the differences $f(z_n) - f(\zeta)$ are all defined and such that $a = \lim_{n\to\infty} (f(z_n) - f(\zeta))(z_n - \zeta)^{-1}$ if $\zeta \neq \infty$, and $a = \lim_{n\to\infty} z_n(f(z_n) - f(\zeta))$ if $\zeta = \infty$. The set of all derivative numbers for $f(z)$ at ζ is denoted by $\mathfrak{D}_f(\zeta)$. The author states without proof the following analogue for the Denjoy theorem on the derivates of a function of a real variable. Theorem 1: At almost every point $\zeta \in E$, one of the following is true: (1) $\mathfrak{D}_f(\zeta)$ is the circle $|z - a(\zeta)| = r(\zeta)$, $0 \leq r(\zeta) < \infty$; (2) $\mathfrak{D}_f(\zeta)$ is the extended complex plane; (3) $\mathfrak{D}_f(\zeta)$ consists of the circle $|z - a(\zeta)| = r(\zeta)$, $0 \leq r(\zeta) < \infty$, and the point $z = \infty$; (4) $\mathfrak{D}_f(\zeta)$ consists of the point $z = \infty$.

Next let E be a set in the finite complex plane and $f(z)$

a finite complex-valued function defined on E. $f(z)$ is said to have k Vallée-Poussin derivatives at $\zeta \in E$ (taken with respect to E) if ζ is a limit point of E and if

$$f(\zeta+h) = f(\zeta)+hf_{(1)}(\zeta)+\frac{h^2}{2!}f_{(2)}(\zeta)+\cdots+\frac{h^k}{k!}f_{(k)}(\zeta)+o(h^k)$$

for all $\zeta+h \in E$. The number $f_{(i)}(\zeta)$ so defined is called the ith Vallée-Poussin derivative. Now let $R_n[f] = \inf_\phi (\sup_{x\in E}|f(x)-\phi(x)|)$, where $\phi(x)$ runs through all rational functions of order not greater than n, and let $R_n(x)$ denote one of the rational functions which realizes this best approximation. Theorem 2: If $R_n[f] \leq C/n^{p+\varepsilon}$ where $C<\infty$ does not depend on n, p is an integer and $\varepsilon>0$, then $f(x)$ has p Vallée-Poussin derivatives with $f_{(i)}(x) = \lim_{k\to\infty} R_{n_k}^{(i)}(x)$ almost everywhere in E, whatever the lacunary sequence $\{n_k\}$. The "almost everywhere" refers to plane measure unless E lies in the real axis, in which case "almost everywhere" refers to linear measure.

A number of other similar results are given, all without proof. {The word "entire" should be omitted from the statement of Theorem 3 in the translation.}

F. W. Gehring (Ann Arbor, Mich.)

9577:

Bogdănescu, V. On some properties of functions of a complex variable. Lucrările Inst. Petrol Gaze Geol. Bucureşti **5** (1959), 359–371. (Romanian. Russian and English summaries)

Contribution to the theory of polygenic functions. The areolar derivative of a function $f(z)=u(x, y)+iv(x, y)$ is defined by

$$\frac{Df(z)}{D\omega} = \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right)+i\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right].$$

A derivation is given of a formula of D. Pompeiu: $\lim (2iA)^{-1}\int_C f(z)dz = Df(z)/D\omega$. Here f is defined in a domain D, C is a convex curve lying with its interior in D, A is the area bounded by C, and the limit is taken as C approaches a point z in its interior. Another such formula of Pompeiu is demonstrated and various generalizations of the classic Cauchy theory are derived.

E. R. Lorch (New York)

9578:

Belinskiĭ, P. P. Normality of families of quasiconformal mappings. Dokl. Akad. Nauk SSSR **128** (1959), 651–652. (Russian)

The author proves the following theorem. Let $\mathfrak{M}$ be a family of functions $\{w=f(z)\}$ possessing the following properties: (1) the functions $f(z)$ are defined in regions and the corresponding mappings $w=f(z)$ are homeomorphisms; (2) auxiliary mappings of the form $z'=az+b$, $w'=aw+b$, carried out on functions in $\mathfrak{M}$, lead again to functions in $\mathfrak{M}$; (3) every sequence of functions $w=f_n(z)$ in $\mathfrak{M}$, defined on regions D_n with a non-empty kernel D, can be transformed into a compact sequence with the aid of normalizations of the form $w'=a_nf_n(z)+b_n$. Here, compactness is understood in the sense of uniform convergence in D to a mapping of the family. Then, the mappings defined by functions of the family are quasiconformal with dilatation not exceeding some number q.

W. Seidel (Notre Dame, Ind.)

9579:

Ikoma, Kazuo. Note on the distortions in certain quasiconformal mappings. Japan. J. Math. **29** (1959), 9–12.

Verf. betrachtet folgende zwei Klassen von K-quasikonformen Homöomorphismen $w=f(z)$ des Einheitskreises $|z|<1$ in die w-Ebene, die den Nullpunkt festlassen: die Klasse S_k ist definiert durch die Bedingung $|f(z)| \sim |z|^k$, $z\to 0$, und die Klasse $S_{1/k}$ durch die Bedingung $|f(z)| \sim |z|^{1/k}$, $z\to 0$. Er zeigt: (1) Weder S_k noch $S_{1/k}$ ist eine normale Familie. (2) Trotzdem besitzt $M(r, f)=\max_\varphi |f(re^{i\varphi})|$, r fest, $0<r<1$, in S_k eine endliche obere Grenze, auf Grund einer von Hersch und Pfluger gegebenen Abschätzung [C. R. Acad. Sci. Paris **234** (1952), 43–45; MR **13**, 736].

{Bemerkung des Ref.: Zum Beweis von (1) genügt es zu zeigen, dass unter den Bildgebieten der Abbildungen in $S_{1/k}$ beliebig grosse Kreise $|w|<R$ vorkommen. Durch

$$f_\rho(z) = z|z|^{\varepsilon-1}, \quad |z| < \rho,$$
$$= z\cdot\rho^{\varepsilon-1}, \quad \rho \leq |z| < 1,$$

$\varepsilon = 1/k$, sind solche Abbildungen definiert.}

A. Pfluger (Zürich)

9580:

Mandelbrojt, Szolem. Quelques théorèmes d'unicité. C. R. Acad. Sci. Paris **250** (1960), 2660–2662.

Le résultat essentiel est le suivant (th. III, énoncé simplifié): soit f une fonction continue sur $[-1, +1]$, nulle en x_0, $\neq 0$; soit $\mathscr{E}_n$ une suite tendant vers zéro et majorant E_n, meilleure approximation polynomiale d'ordre n de f; soit enfin $C(h)$ une fonction strictement croissante majorant $|f(x_0+h)|$ $(0<h\leq 1-x_0)$, et $c(t)$ sa fonction inverse; alors $c(\mathscr{E}_n)=o(\mathscr{E}_n^{1/n})$ $(n\to\infty)$. De là on tire (th. II), pour les fonctions F continues 2π-périodiques $\sim\sum a_n \cos n\theta$, un résultat analogue faisant intervenir $\sum_{m>n}|a_m| = r_n$ au lieu de la meilleure approximation polynomiale. Des cas particuliers (th. I et IV) sont les suivants: si $F(\theta_0+h)=O(\exp(-h^{-\rho}))$ et si $\liminf[(n\log n)^{2/\rho}r_n^{1/n}] < \infty$, on a $F\equiv 0$; si $f(x_0+h)=O(\exp(-h^{-\rho}))$ et si $\liminf[(n\log n)^{1/\rho}E_n^{1/n}]<\infty$, on a $f\equiv 0$. Enfin (th. V), si les suites p_n et q_n satisfont $0\leq q_n\leq n<p_n$ et que la classe $C\{M_n/[(p/n)-1]^{pn-q_n}\}$ est quasi-analytique, les inégalités $|f^{(n)}(x)x^{p_n}\log^{q_n}x| \leq M_n$ $(1\leq a<x<\infty)$ entrainent $f\equiv 0$.

J.-P. Kahane (Montpellier)

9581:

Mihal'čuk, V. G. The Cauchy integral and the Riemann-Roch theorem for quasi-analytic functions on Riemann surfaces. Issledovaniya po sovremennym problemam teorii funkciĭ kompleksnogo peremennogo, pp. 425–436. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960. (Russian)

The paper concerns extensions of classical theorems of function theory to solutions of various types of elliptic systems of partial differential equations. In particular, the Riemann-Roch theorem is extended to systems which, in complex notation, have form $\partial w/\partial\bar{z} = B(z)\overline{\partial w/\partial z}$. Similar results have been obtained by Bers [*Contributions to the theory of Riemann surfaces*, pp. 157–165, Princeton Univ. Press, Princeton, N.J., 1953; MR **15**, 304].

W. Kaplan (Ann Arbor, Mich.)

9582:

Rivlin, T. J. On the maximum modulus of polynomials. Amer. Math. Monthly **67** (1960), 251–253.

Let $p(z)$ be a polynomial of degree n and let $M(p, r)$ denote the maximum of $|p(z)|$ on $|z|=r$. The author considers the class Q of polynomials having no zeros inside the unit circle and normalized in such a way that $M(p, 1)=1$. It was shown previously by N. C. Ankeny and the author [Pacific J. Math. **5** (1955), 849–852; MR **17**, 833] that $M(p, R)\leq(1+R^n)/2$ for $p\in Q$ and $R\geq 1$. In the present note, it is proved that $M(p, r)\geq((1+r)/2)^n$ for $p\in Q$ and $r\leq 1$, the lower bound being attained by $p(z)=((\lambda+\mu z)/2)^n$, where $|\lambda|=|\mu|=1$. From these two results, upper and lower bounds for $M(p, \rho)$ are obtained for all $\rho\geq 0$ and for those $p\in Q$ which have all their zeros on $|z|=1$. *F. Herzog* (E. Lansing, Mich.)

9583:

Parodi, Maurice. À propos de la localisation des zéros des polynomes. C. R. Acad. Sci. Paris **251** (1960), 1851–1852.

The author states without proof that the method of his earlier paper [same C. R. **247** (1958), 391–393, 908–910; MR **20** #4633; **21** #127] may be applied to the polynomial $f(z)=z^n+a_1z^{n-1}+a_2z^{n-2}+\cdots+a_n$ to show that all its zeros lie in the union of the regions defined by the inequalities: $|z|\leq 1$, $|z^2+a_1z+a_2|\leq|a_3|+\cdots+|a_n|$.

M. Marden (Milwaukee, Wis.)

9584:

Parodi, Maurice. Sur la localisation des racines des équations réciproques. Bull. Sci. Math. (2) **83** (1959), 21–23.

Transform the reciprocal equation (*) $x^{2n}+a_1x^{2n-1}+\cdots+a_{2n-1}x+a_{2n}=0$ where $a_{2n-k}=a_k$, $k=0, 1, \cdots, n$, by setting $z=x+(1/x)$, into the equation (**) $V_n(z)+a_1V_{n-1}(z)+\cdots+a_{n-1}V_1(z)+a_n=0$ where $V_n(z)-zV_{n-1}(z)+V_{n-2}(z)=0$ with $V_0(z)=2$ and $V_1(z)=z$. On writing the result of eliminating the $V_k(z)$ in the form of a certain determinant D and applying Hadamard's theorem on the non-vanishing of a determinant, the author proves that no root of (**) can satisfy the inequality $|z|>3$ if $|a_1|+|a_2|+\cdots+|a_n|<1$. *M. Marden* (Milwaukee, Wis.)

9585:

Jankowski, W. Sur les zéros d'un polynôme contenant des paramètres arbitraires. Ann. Polon. Math. **8** (1960), 33–53.

The author completes the proof that for arbitrary complex a and b the polynomial $F(z)=(z+P)^p+az^{p+1}+bz^{p+2}$ has at least p zeros in the disk $|z|\leq R|P|$, where $R=\frac{1}{4}[2(p+1)+(2p^2+2p)^{1/2}]$. He had proved this theorem in the cases $|a/b|\leq R$ and $|a/b|\geq R^2/(p+1)$, and for P and a/b real [Ann. Univ. Mariae Curie-Skłodowska Sect. A **5** (1951), 31–92; MR **15**, 120]. There had remained the case $R<a/b<R^2/(p+1)$ treated in the present paper by a lengthy, careful study of the variation of $\arg F(z)$ on the circle $|z|=R|P|$. *M. Marden* (Milwaukee, Wis.)

9586:

Specht, Wilhelm. Die Lage der Nullstellen eines Polynoms. IV. Math. Nachr. **21** (1960), 201–222.

[For part III see Math. Nachr. **16** (1957), 369–389; MR **20** #2427.] Let $\{\psi_k(x)\}$ form a set of orthonormal polynomials on $-\infty<x<\infty$ with respect to a positive weight function $h(x)$ and let λ_kz^k be the highest degree term in $\psi_k(x)$. In a previous paper [W. Specht, ibid. **15** (1956), 353–374; MR **19**, 400] each zero Z of a given polynomial

$$f(z)=a_0+a_1z+\cdots+a_nz^n=b_0\psi_0(z)+b_1\psi_1(z)+\cdots+b_n\psi_n(z)$$

was shown to satisfy the inequality

$$|\mathrm{Im}(Z)|^2\leq R_n^2(b_0^2+b_1^2+\cdots+b_{n-1}^2).$$

This bound has been improved in the present paper to read

$$4|\mathrm{Im}[Z+(1/2)R_nb_{n-1}]|^2\leq R_n^2\{b_0^2+b_1^2+\cdots+b_{n-2}^2+[\mathrm{Im}(b_{n-1})]^2\}.$$

Also certain circles are specified for the location of each zero Z of $f(z)$. The results are obtained by introducing matrices for which $f(z)$ is the characteristic polynomial and then using various known theorems on the location of the characteristic roots of a matrix.

M. Marden (Milwaukee, Wis.)

9587:

Heffter, Lothar. ★Begründung der Funktionentheorie auf alten und neuen Wegen. 2te wesentlich verbesserte Aufl. Springer - Verlag, Berlin - Göttingen - Heidelberg, 1960. viii + 64 pp. DM 19.80.

This booklet is devoted to a presentation of various ways, starting from different assumptions, that can be followed to establish that a complex-valued function $f(z)$ of a complex variable has a power series expansion. The contents are essentially the same as those of the first edition [1955; MR **16**, 807].

W. J. Thron (Boulder, Colo.)

9588:

Srivastava, K. N. On an entire function defined by a gap Dirichlet series. Math. Student **27** (1959), 25–36.

Suppose λ_n ($n=1, 2, \cdots$) is a positive increasing sequence, with $\liminf(\lambda_{n+1}-\lambda_n)>0$, and that $f(s)=\sum_0^\infty a_ne^{\lambda_ns}$ is an integral function convergent for all s. Put $M(\sigma, f^{(p)}(s))=\sup_{-\infty<t<\infty}|f^{(p)}(\sigma+it)|$, let $N(\sigma, f)$ be the largest index for which $\mu(\sigma, f)=\mathrm{Max}_n|a_n|e^{\lambda_n\sigma}$ is attained, and $\lambda_N=\lambda_{N(\sigma,f)}$. The following and related results are proved. Theorem 1: If $f(s)$ has infinite Ritt order and $p=p(\lambda_N)=o(\lambda_N/\log\lambda_N)$, then

$$\liminf_{\sigma\to\infty}\frac{\log M(\sigma, f^{(p)})}{\lambda_{N(\sigma,f)}}=0.$$

Theorem 5: If $\alpha>0$ and $[\lambda_{n+1}/\lambda_n]^\alpha|a_n/a_{n+1}|$ ultimately increases, then

$$\limsup\frac{\log M(\sigma, f)}{\log\mu(\sigma, f)}\leq 1+\alpha^{-1}.$$

The paper contains numerous minor errors.

M. E. Noble (Nottingham)

9589:

Korobeĭnik, Yu. F. Some properties of series of functions. Uspehi Mat. Nauk **15** (1960), no. 4 (94), 149–156. (Russian)

Existence of zeros is shown for the partial sums of a series of analytic functions near the boundary of the region of convergence. Let $S_n(z)$ be the nth partial sum of the series $\sum a_n\varphi_n(z)$, where $\varphi_n(z)$, $n=0, 1, 2, \cdots$, are analytic functions satisfying conditions of the form $\limsup |\varphi_n(z)|^{1/n}=u(z)$ in a region $R_1<u(z)<R_2$, and where $\limsup |a_n|^{1/n}=1/\rho_0$ and $R_1<\rho_0<R_2$. Let $\mu_n(z_0, \delta)$ be the number of zeros of $S_n(z)$ in the disk $|z-z_0|\leq\delta$. The conclusion $\limsup \mu_n(z_0, \delta)/n>0$ is obtained from the hypothesis $u(z_0)=\rho_0$ and a condition of linear independence on the functions $\varphi_n(z)$. Results on the overconvergence and non-continuability of power series are observed to generalize, with little change in proofs, to the series here considered. *L. de Branges* (Wayne, Pa.)

9590:

U, Šo-mo [Ou, Šo-mo]. Some properties of functions which are analytic in a circle. Advancement in Math. **3** (1957), 250–256. (Chinese. Russian summary)

Let $f(z)=\sum_0^\infty a_nz^n$ be analytic in the unit circle $|z|<1$. If

$$\|f(z)\|_p=\left\{\int_{-\pi}^{\pi}|f(re^{i\theta})|^p\,d\theta\right\}^{1/p}<\infty \qquad (0<r<1, p>0),$$

then $f(z)$ is said to be of the class H_p. In this note, the mean modulus $\|f(z)\|_p$ and the boundary values of the functions of the class H_p are discussed.

Let $\sigma_n(z)=\sum_{\nu=0}^n s_\nu(z)/(n+1)$, where $s_n(z)=\sum_0^n a_kz^k$. The author proves the following theorem. If $f\in H_p$, $\|f(z)\|_p\leq B$ $(0\leq r<1)$, then $\|\sigma_n(z)\|_p\leq B$, $\|s_n(\frac{1}{2}z)\|_p\leq B$ $(0\leq r<1$; $n=0, 1, 2, \cdots)$. Conversely, if $\|\sigma_n(z)\|\leq B$, then $\|f(z)\|_p\leq B$, where B is an absolute constant. Other theorems concerning similar properties are also discussed. *F. C. Hsiang* (Taipeh)

9591:

Varganova, S. V.; Korobeĭnik, Yu. F. Over-convergence and non-continuability of series of functions. Dokl. Akad. Nauk SSSR **137** (1961), 499–501 (Russian); translated as Soviet Math. Dokl. **2**, 290–292.

Let $u(z)$ be a non-negative continuous function such that the equations $u(z)=\rho$ represent rectifiable curves C_ρ which shrink toward the origin as $\rho\to0$. Let $\{\phi_n(z)\}$ be a system of functions that resembles, in a specified sense, the system $\{(u(z))^n\}$. For functions $f(z)=\sum a_n\phi_n(z)$ which are holomorphic in the interior of a curve C_ρ, the authors state theorems analogous to Ostrowski's direct theorem on gap series. For their analogues of Ostrowski's converse theorem, they invoke the additional hypothesis that the function $\log u(z)$ is superharmonic. *G. Piranian* (Ann Arbor, Mich.)

9592:

Vinogradov, V. S. Concerning a method of solution of the Poincaré problem for analytic functions. Dokl. Akad. Nauk SSSR **132** (1960), 17–19 (Russian); translated as Soviet Math. Dokl. **1**, 455–457.

The problem is the following: Find a function f, analytic in $|z|<1$ and satisfying on $|z|=1$ the condition $\mathrm{Re}\{a(z)f'(z)+b(z)f(z)\}=c(z)$ where a, b, c are defined and satisfy a Hölder condition on $|z|=1$ and $a\neq0$. Let n be the index of a; p the function analytic in $|z|<1$ satisfying $\mathrm{Im}\,p(z)=\arg[z^{-n}a(z)]$ on $|z|=1$; and $\gamma=be^{\mathrm{Re}\,p}$ on $|z|=1$. The basic assumption is that the Fourier series of γ contains only finitely many terms with negative powers. The condition on f can then be written

$$\mathrm{Re}\left\{\frac{z^ke^{p(z)}f'(z)+Q(z)f(z)}{z^l}\right\}=c_1(z)$$

on $|z|=1$, where k and l are non-negative integers and Q is analytic. It is easy to find the most general possible expression for $z^ke^{p(z)}f'(z)+Q(z)f(z)$, and f is then obtained by solving a first-order differential equation. *H. Widom* (Ithaca, N.Y.)

9593:

Tarumoto, Kôichi. Cauchy integral on Riemann surfaces. Proc. Japan Acad. **35** (1959), 349–353.

On a Riemann surface W, let $dN(\zeta, z)$ be an analytic differential with respect to ζ and an analytic function with respect to z except for $z=\zeta$ where there is a pole of residue 1. If Γ is a compact curve and f is a function defined on Γ, set $L(f)=(2\pi i)^{-1}\int f(\zeta)dN(\zeta, z)$. The author discusses the behavior of $L(f)$ as the point z crosses Γ and claims to give a new proof of Read's version of the converse of the Cauchy theorem [Acta. Math. **100** (1958), 1–22; MR **20** #4640]. The reviewer is unable to follow the author's extension of the operator L to differentials, and it seems to the reviewer that he tacitly assumes the existence of a global uniformizer. *H. L. Royden* (Stanford, Calif.)

9594:

Maschler, Michael. Analytic functions of the classes L^2 and l^2 and their kernel functions. Rend. Circ. Mat. Palermo (2) **8** (1959), 163–177.

This paper deals with the characterization of multiply-connected domains D in the complex plane by means of the Bergman kernel function. Let $L^{(2)}(D)$ be the class of all regular analytic and square-integrable functions in D, $l^{(2)}(D)$ be the subclass of $L^{(2)}(D)$ of functions with a single-valued integral in D; denote by $K(z, t)$ and $\bar{K}(z, t)$ the kernel functions of the classes $L^{(2)}(D)$ and $l^{(2)}(D)$, respectively. The main result is: If there exists a point $t\in D$ such that $K(z, t)=\bar{K}(z, t)$ identically in z, then the domain D has at most one boundary component which is not completely point-like in the sense of Grötzsch. A set $\Gamma\in D$ is called K- or G-removable, if its deletion from D leads to a domain with unchanged kernel or Green's function, respectively. It is shown that Γ is K-removable if and only if it is G-removable. Consider the orthogonal decomposition $L^{(2)}(D)=l^{(2)}(D)\oplus h^{(2)}(D)$. If D has finite connectivity, it is well-known that $h^{(2)}(D)$ is spanned by the derivatives of the harmonic measures of D. Let $f(z)$ map D onto a circular ring slit along concentric circular arcs; the logarithmic derivative $f'(z)/f(z)$ lies in $h^{(2)}(D)$ since $\log|f(z)|$ is easily constructed from the harmonic measures of D. It is shown that an analogous result holds also in the case of infinite connectivity. Finally, the author discusses the difference between representative and minimal domains (as defined by Bergman) in the case of higher connectivity. *M. Schiffer* (Stanford, Calif.)

9595:

Ou, Šo-mo. Quelques théorèmes de fermeture. Advancement in Math. **3** (1957), 246–249. (Chinese. French summary)

Let $z=z(t)$ be a function which transforms conformally the unit circle $|t|<1$ into the domain D, and suppose that there exist two absolute constants m and M such that $0<m\leq|z'(t)|\leq M(|t|<1)$. The author defines two classes $H_p(D)$ and $h_p(D)$ on D of analytic functions $f(z)$ and the harmonic functions $U(z)$ respectively, such that

$$\|f(z)\| = \left\{\iint_D |f|^p\,dx\,dy\right\}^{1/p} < \infty \qquad (p>0, z=x+iy),$$

$$\|U(z)\| = \left\{\iint_D |U|^p\,dx\,dy\right\}^{1/p} < \infty.$$

In this note, the author proves the following two theorems.

Theorem 1: (i) If $f(z)=U(z)+iV(z)$ is analytic in D and $U(z)\in h_p(D)$ $(p>1)$, then $f(z)\in H_p(D)$. (ii) If, for the sequence of functions $f_n=U_n+iV_n$, $U_n\in h_p(D)$ and $\|U_n(z)-U(z)\|\to 0$ $(n\to\infty)$, and if at the point $z_0=z(0)\in D$, $\lim_{n\to\infty}\operatorname{Im} f_n(z_0)=\operatorname{Im} f(z_0)$, then

$$\lim_{n\to\infty}\|f_n(z)-f(z)\| = 0$$

and

$$\|f_n(z)-f(z)\| \leq (M/m)^{2/p}(1+A_p)\|U_n(z)-U(z)\| + \{\text{meas } D\}^{1/p}|\operatorname{Im}(f_n(z_0)-f(z_0))|,$$

where m and M are the constants given previously and A_p is the constant of M. Riesz [Math. Z. **27** (1927), 218–244].

Theorem 2: If $V(z_0)=0$, then

$$\|f(z)\| \leq (M/m)^{2/p}(1+A_p)\|U(z)\| \quad (p>1).$$

There are two analogous classes E_p and e_p of functions which possess similar properties. *F. C. Hsiang* (Taipeh)

9596:

Ozawa, Mitsuru. A set of capacity zero and the equation $\Delta u=Pu$. Kōdai Math. Sem. Rep. **12** (1960), 76–81.

The author discusses the classification of open Riemann surfaces with respect to the existence of various classes of solutions of the equation $\Delta u=Pu$. He is primarily concerned with giving a new treatment of results previously stated by himself, L. Myrberg and the reviewer. These include the fact that a bounded solution of $\Delta u=Pu$ in the exterior of a set E of capacity zero can have only removable singularities on E. *H. L. Royden* (Stanford, Calif.)

9597:

Gál, I. S. Conformally invariant metrics and uniform structures. I, II. Nederl. Akad. Wetensch. Proc. Ser. A **63**=Indag. Math. **22** (1960), 218–231, 232–244.

Let x and y be two points on a Riemann surface. The author defines two conformally invariant metrics by letting $d_2(x, y)$ be the square root of the extremal length of the class of all Jordan curves which bound simply connected regions containing x and y, and let $d_1(x, y)$ be the corresponding quantity when Jordan arcs as well as curves are allowed. The author proves that these are metrics which (except in certain simply connected cases) are compatible with the topology of the surface. The surface is complete with respect to the metric d_1, and this metric is related to Bergman's invariant Riemannian metric. The completion of the surface with respect to the metric d_1 is closely connected with the Carathéodory-Ahlfors theory of prime ends. The author derives a number of theorems which generalize (and give new proofs of) the Carathéodory theory of boundary correspondence for Jordan domains in the plane.

H. L. Royden (Stanford, Calif.)

9598:

Accola, Robert D. M. Differentials and extremal length on Riemann surfaces. Proc. Nat. Acad. Sci. U.S.A. **46** (1960), 540–543.

Let W be a Riemann surface and c a homology class of curves. Let $\lambda(c)$ be the extremal length of the class and let $\sigma(c)$ be the unique real differential in Γ_{h0} [see Ahlfors and Sario, *Riemann surfaces*, Princeton Univ. Press, Princeton, N.J., 1960; MR **22** #5729] such that for any square-integrable harmonic differential ω we have $\iint \omega\wedge\sigma(c)=\int_c\omega$. The author establishes the following. Theorem 1: If W is an arbitrary surface, $\|\sigma(c)\|^2=\lambda(c)$. Theorem 2: Let W and W' be homeomorphic surfaces. Let c and c' be corresponding cycles. Then the following two conditions are equivalent: (1) there exists $M\geq 1$ such that $M^{-1}\lambda(c)\leq\lambda(c')\leq M\lambda(c)$; (2) there exists ϕ, a continuous linear isomorphism from $\Gamma_{h0}(W)$ onto $\Gamma_{h0}(W')$ such that $\phi\sigma(c)=\sigma(c')$ and $\int_c\sigma=\int_{c'}\phi\sigma$ for all c, c'; Theorem 4: Two closed surfaces, W and W', of the same genus are conformally equivalent if and only if $\lambda(c)=\lambda(c')$ for corresponding cycles. *H. L. Royden* (Stanford, Calif.)

9599:

Blatter, Christian. Une inégalité de géométrie différentielle. C. R. Acad. Sci. Paris **250** (1960), 1167.

Let F be a closed surface of genus $g\geq 1$. Let A be its area and L the minimum length of a cycle which is not homologous to 0. It is a conjecture of long standing that $L^2/A\leq\sigma_g$ where σ_g depends only on g. The author announces a proof which is based partly on an inequality of Minkowski for quadratic forms, partly on an explicit determination of the extremal length $\lambda(Z)$ of cycles that belong to a given homology class Z. If ω is the reproducing differential associated with Z, then $\lambda(Z)=\|\omega\|^2$. Under more general conditions this result has also been proved by R. Accola [#9598].

The reviewer was unable to reconstruct the proof on the basis of the outline. The author promises to publish a complete proof. *L. Ahlfors* (Cambridge, Mass.)

9600:

Clunie, J. On the maximum modulus of an integral function. Proc. London Math. Soc. (3) **10** (1960), 161–179.

Let $f(z)=\sum a_nz^n$ be an integral (entire) function and let $M(r)$ denote its maximum modulus on the circle $|z|=r$. The paper is concerned with the question of how small $M(r)$ may become as compared to $g(r)=\sum|a_n|r^n$, either for sufficiently large r or for arbitrarily large r. In

the case of sufficiently large r, the quotient $Q(r)=M(r)[\log M(r)]^{1/4}/g(r)$ is used as a basis for this comparison, and it is shown that, if the lower order of $f(z)$ is ρ, $0<\rho<\infty$, then $\limsup Q(r)\geqq c_1\rho^{-1/2}$, where c_1 is a positive numerical constant. On the other hand, for any given ρ with $0<\rho<\infty$, there exists an integral function $f(z)$ of lower order ρ for which $\limsup Q(r)\leqq c_2\rho^{-1/2}$, c_2 again being a certain numerical constant. In the case of arbitrarily large r, the following result is shown. For any $\phi(r)\to 0$, there exists an integral function $f(z)$ of given order ρ, $0\leqq\rho<\infty$, for which $\liminf[M(r)]^{1+\phi(r)}/g(r)=0$. In both cases, results concerning $\rho=0$ and $\rho=\infty$ are also obtained. *F. Herzog* (E. Lansing, Mich.)

9601:

Gol'dberg, A. A. Distribution of values of meromorphic functions with separated zeroes and poles. Dokl. Akad. Nauk SSSR **137** (1961), 1030–1033 (Russian); translated as Soviet Math. Dokl. **2**, 389–392.

Let p be a positive integer and let $D(\eta,\phi)$ be the union of the p sectors $|\arg z-\phi-2k\pi/p|\leqq\eta$ $(k=0,1,2,\cdots,p-1)$ in the complex z-plane. Let $f(z)$ be a function meromorphic in $|z|<R\leqq\infty$, with zeros $a_1, a_2, \cdots$, poles $b_1, b_2, \cdots$. Suppose that there is a positive integer p, a real ϕ and an η, $0\leqq\eta<\pi/2p$, such that $\sum|a|^{-p}$ extended over all zeros of $f(z)$ outside $D(\phi,\eta)$ is convergent and $\sum|b|^{-p}$ extended over all poles outside $D(\phi+\pi/p,\eta)$ is convergent. Then (in the standard notations of the Nevanlinna theory): (a) $R=\infty$, $0<\limsup T(r,f)r^{-p}<\infty$ implies $\delta(0,f)=\delta(\infty,f)=1$; (b) $R=\infty$ and $\limsup T(r,f)r^{-p}=\infty$ or $R<\infty$ and $\lim_{r\to R}T(r,f)=\infty$ imply

$$\liminf\{m(r,0)+m(r,\infty)\}/T(r)\geqq 2\cos p\eta/(1+\cos p\eta).$$

An example is given to show that in case (b) it can happen that $\delta(0,f)=\delta(\infty,f)=0$. The author conjectures that in case (b) with $\eta=0$, $\delta(0,\beta)>0$, $\delta(\infty,f)>0$ and proves a result in this direction. He also mentions the following result: If $g(z)$ is an entire function of finite integral order p, all of whose zeros lie in a sector of vertex 0 and opening strictly less than $\pi/(p+1)$ (p odd) or $(1-(2p)^{-1})\pi/(p+1)$ (p even), then $\delta(0,g)>0$.

The principal tool in the proofs is the formula

$$(2\pi)^{-1}\int_0^{2\pi}e^{-ip\theta}\log|f(re^{i\theta})|\,d\theta=(2p)^{-1}\sum_{|a|<r}(r^pa^{-p}-a^{-p}r^{-p})-(2p)^{-1}\sum_{|b|<r}(r^pb^{-p}-b^{-p}r^{-p})+O(r^p)$$

obtained by differentiation of the Poisson-Jensen formula. *W. H. J. Fuchs* (Ithaca, N.Y.)

9602:

Anastassiadis, Jean. ★Recherches algébriques sur le théorème de Picard-Montel. Préface de Paul Montel. Actualités Sci. Ind. No. 1273. Exposés sur la Théorie des Fonctions, XVIII. Hermann, Paris, 1959. 52 pp. 9 NF.

This work deals with entire and meromorphic functions of finite order. In order to illustrate the problems treated and the methods applied, let us consider a non-constant entire function $f(z)=c_0+c_1z+\cdots+c_nz^n+\cdots$ of order p. If $f(z)$ omits a finite value a, then $f(z)-a=e^{P(z)}$, where $P(z)$ is a polynomial of degree p. Hence, $f'(z)=(f(z)-a)P'(z)$, and a comparison of the coefficients of f and P' yields the following result. If

$$A_n=\begin{vmatrix}c_n & c_{n+1} & \cdots & c_{n+p-1}\\ \vdots & \vdots & & \vdots\\ c_{n+p-1} & c_{n+p} & \cdots & c_{n+2p-2}\end{vmatrix}\neq 0,$$

then the exceptional value a is uniquely determined by the $n+2p$ first coefficients of $f(z)$ and depends on them rationally. In particular, if $A_1\neq 0$, the exceptional value is a rational function of the $2p+1$ first coefficients of $f(z)$.

This simple way of reasoning can be extended to far more general cases. For entire functions, similar results are derived in the cases where a value a is taken only of finite number of times or where $f(z)$ possesses an exceptional polynomial $M(z)$ such that $f(z)-M(z)$ is non-zero. The method also applies to meromorphic functions of finite order and to linear combinations of entire functions.

Finally, defining $\mathcal{E}(x)$ to be the point set in which an entire function $f(z)$ takes the value x, account being taken of the possible multiplicity of the roots of the equation $f(z)-x=0$, the author establishes the following uniqueness theorem. An entire function of finite order q is uniquely determined by its $[q]+1$ first Taylor coefficients c_i $(i=0,1,\cdots,[q])$ and by a point set $\mathcal{E}(x)$, where $x\neq c_0$.

O. Lehto (Helsinki)

9603:

Pogorzelski, W. Problèmes aux limites discontinus dans la théorie des fonctions analytiques. J. Math. Mech. **9** (1960), 583–606.

A problem is to determine a system of holomorphic functions in a set of domains whose limit values on the boundary satisfy given relations. The case where the limit functions admit points of discontinuity on the boundary has been studied by N. I. Muskhelishvili [*Singular integral equations*, Noordhoff, Groningen, 1953; MR **15**, 434]. The author studies problems connected with discontinuous limits, both linear and nonlinear, under more general hypotheses for the limit functions and boundary. The details are too technical to state here.

M. S. Robertson (New Brunswick, N.J.)

9604:

Heins, Maurice. On the boundary behavior of a conformal map of the open unit disk into a Riemann surface. J. Math. Mech. **9** (1960), 573–581.

The object of the paper is to extend a theorem of Collingwood and Cartwright [Acta Math. **87** (1952), 83–146; MR **14**, 260] and a theorem of Carathéodory [Comment. Math. Helv. **19** (1946), 263–278; MR **8**, 508] concerning the boundary behaviour of meromorphic functions to the case of a conformal map of $|z|<1$ into an arbitrary Riemann surface.

If φ denotes a conformal map of $|z|<1$ into a Riemann surface S, a point ζ on $|z|=1$ is called a Fatou point of φ if $\varphi(z)$ tends to a point of S or else $\varphi(z)$ tends to the ideal boundary of S as $z\to\zeta$ sectorially; for a function $f(z)$ meromorphic in $|z|<1$, a complex number α (including ∞) is called a Fatou value at ζ if α is a sectorial limit of $f(z)$ at ζ. The range of φ at ζ is the set of points w for which ζ is an accumulation point of $\varphi^{-1}(w)$; for a meromorphic function $f(z)$ the point ζ is called a Picard point of f if the range of f at ζ omits at most two points of the extended

plane, while for the more general map φ, the point ζ is called a Picard point of φ if one of the following conditions is satisfied: (1) whenever S is conformally equivalent to the extended plane, the range of φ at ζ omits at most two points of S; (2) whenever S is conformally equivalent to the finite plane, the range of φ omits at most one point of S; (3) In all other cases, the range of φ at ζ is S. For f meromorphic in $|z|<1$, Collingwood and Cartwright have shown that every point of $|z|=1$ is either a Picard point or else is in the derived set of the set of Fatou points; the present author shows that the same statement holds if the definition of a Picard point is modified as indicated above.

Let $\mathcal{O}$ denote the open set of points $s \in S$ such that there exist an open set Ω, $s \in \Omega$, and a positive number ρ with the property that, for all η with $|\eta|=1$ and $|\eta-\zeta|<\rho$, either $\varphi(r\eta)$ tends to a point of $S-\Omega$ as $r\to 1$ or else $\varphi(r\eta)$ tends to the ideal boundary of S as $r\to 1$. Let $\mathcal{O}_1$ denote the union of those components of $\mathcal{O}$ having a non-empty intersection with the cluster set $C(\varphi, \zeta)$ of φ at ζ. The author shows that Carathéodory's result that $C(\varphi, \zeta)=(S-\mathcal{O}) \cup \mathcal{O}_1$ still holds in the case that S is an arbitrary Riemann surface, rather than the extended plane.

A. J. Lohwater (Houston, Tex.)

9605:

Havinson, S. Ya. The analytic capacity of plane sets, some classes of analytic functions and the extremum function in Schwarz's lemma for arbitrary regions. Dokl. Akad. Nauk SSSR **128** (1959), 896–898. (Russian)

Let Γ be a bounded closed plane set and let G be the component of its complement relative to the extended plane which contains the point ∞. Let K be a class of functions analytic on G. The analytic capacity of Γ relative to K and the point z_0 of G is defined as $\sup_{f\in K} \lim_{z\to\infty} |zf(z)|$, if $z_0=\infty$, and is defined correspondingly if $z_0 \neq \infty$. If $z_0=\infty$ and if K is the class of functions analytic on G of modulus ≤ 1, this supremum is the analytic capacity of Ahlfors [Duke Math. J. **14** (1947), 1–11; MR **9**, 24] and is denoted below by $\Omega(\Gamma)$. Let f^* be a function on G maximizing $|f'(\infty)|$ for f in the class K just described. Then f^* is uniquely determined up to a multiplicative constant of modulus 1.

Furthermore

$$(*) \qquad \Omega(\Gamma) = (2\pi)^{-1} \inf \int_\gamma |\Phi(z)|\, ds,$$

where the infimum is taken for γ any rectifiable curve enclosing Γ and Φ is any function regular in the unbounded domain bounded by γ, equal to $1+O(1/z)$ near ∞. Necessary and sufficient conditions are found on a function f^* that it be extremal, involving the extremal problem (*). *J. L. Doob* (Urbana, Ill.)

9606:

Havinson, S. Ya. The analytic capacity of sets as related to mass distributions. Dokl. Akad. Nauk SSSR **128** (1959), 1129–1131. (Russian)

The notation is that of the preceding review [#9605]. If Γ has finite Painlevé length $l(\Gamma)$, every bounded regular function on G can be represented as a Cauchy-Stieltjes integral (1) $\int_\Gamma d\mu/(\zeta-z)$, where μ is a complex measure supported by the outer boundary of Γ. Moreover, $\operatorname{Var}\mu \leq \sup|f|\, l(\Gamma)$. This result implies that $\Omega(\Gamma)=\max_\mu \operatorname{Var}\mu$ for measures μ such that the integral in (1) has modulus ≤ 1 in G. For an arbitrary G there is an analytic function L with the following properties: $L(z)=1+O(z^{-2})$ near ∞; L does not vanish and $\log L$ is single-valued; $L-1$ can be expressed in the form (1) with some measure μ_L of variation $\leq \Omega(\Gamma)$; if f is analytic on G, of modulus ≤ 1, vanishing at ∞, then fL can be written in the form (1) with $\operatorname{Var}\mu \leq \Omega(\Gamma)$. For a properly normed extremal function f^* the corresponding measure function μ^* is positive, with $\mu^*(\Gamma)=\Omega(\Gamma)$. If G is simply connected, μ^* is proportional to harmonic measure relative to ∞. Other related results are stated. For example, if f is bounded in G and has superior limit (in modulus) at most m almost everywhere (μ^*) on Γ, it follows that $\lim_{z\to\infty}|zf(z)| \leq m\Omega(\Gamma)$. It is stated that the results lead to general settings for classical Lusin-Privalov boundary limit theorems, and related theorems.

J. L. Doob (Urbana, Ill.)

9607:

Vituškin, A. G. Example of a set of positive length but of zero analytic capacity. Dokl. Akad. Nauk SSSR **127** (1959), 246–249. (Russian)

The author constructs an example of a plane set of strictly positive Hausdorff linear measure but of zero analytic capacity. (For the definition of analytic capacity see Ahlfors [Duke Math. J. **14** (1947), 1–11; MR **9**, 24] or Vituškin [Dokl. Akad. Nauk SSSR **123** (1958), 778–781; MR **21** #2056].) *J. L. Doob* (Urbana, Ill.)

9608:

Gal'perin, I. M. On the theory of p-valent spiral functions. Dopovidi Akad. Nauk Ukraïn. RSR **1959**, 1051–1053. (Ukrainian. Russian and English summaries)

The author considers extremal problems for functions of the class $S_p(m)$. These are functions p-valent for $|z|<1$, and of the form $f_m(z)=z^p+b_{p+m}z^{p+m}+\cdots$, satisfying the relation $\operatorname{Re}(e^{-i\theta}zf'(z)[f(z)]^{-1})>0$, where $-\pi/2<\theta<\pi/2$. Using the representation

$$f_m(z)=z^p \exp\left[-pe^{i\theta}(\pi m)^{-1}\int_0^{2\pi} \log(1-z^m e^{-it})\,d\mu(t)\right],$$

where $d\mu$ is a mass distribution on $(0, 2\pi)$ with total mass $2\pi\cos\theta$, he finds, for instance, the extreme value of $|\Phi[\log(f_m(z)/z^p)]|_{|z|=r}$, where $\Phi(w)$ is a given entire function, with $\Phi'(w)\neq 0$. The extremum is achievable only by functions f_m related in a simple way to Koebe functions.

E. Reich (Aarhus)

9609:

Alenicyn, Yu. E.; Havinson, S. Ya. The radius of p-valence for bounded analytic functions in multiply-connected domains. Mat. Sb. (N.S.) **52 (94)** (1960), 653–657. (Russian)

Let G be a multiply-connected domain of finite multiplicity n, and bounded by n non-degenerate continua. Let z_0 be a fixed point in G, and let $B(A_0, A_1)$ be the family of functions $f(z)$ which are analytic in G, with $|f(z)| \leq 1$ in G, and with $f(z_0)=A_0$, $f'(z_0)=A_1$. The authors prove that for each such family there is a radius of p-valence r_p with the property that every function in the family is p-valent in $|z-z_0|<r_p$, but there is at least one function, called an extremal function, which has valence greater

than p in $|z-z_0|<r_p+\varepsilon$ for each $\varepsilon>0$. Further each extremal function maps G onto the unit circle covered m times, where $\max(n, p+1)\leq m\leq n+p+1$.

A. W. Goodman (Lexington, Ky.)

9610:

Kuz'mina, G. V. Numerical determination of radii of univalence of analytic functions. Trudy Mat. Inst. Steklov. **53** (1959), 192–235. (Russian)

If $f(z)$ is an analytic function regular in a disk $|z|<R$, the radius of univalence r $(0\leq r\leq R)$ is defined as the radius of the largest concentric disk in which $f(z)$ is univalent. The author discusses various methods by means of which r can be determined for a given $f(z)$. Rough estimates are obtained from various known theorems concerning univalent functions, and these estimates are then refined—and, in some cases, the exact value of r is found—by appropriate use of the observation that, on the circumference $|z|=r$, the function $f(z)$ has either a critical point, or else there are at least two points on the curve $w=f(re^{i\theta})$, $0\leq\theta<2\pi$, at which the tangents to the curve coincide.

Z. Nehari (Pittsburgh, Pa.)

9611:

Piranian, George. An isolated schlicht function. Abh. Math. Sem. Univ. Hamburg **24** (1960), 236–238.

Let R be the space of functions $f(z)=\sum_1^\infty a_n z^n$ holomorphic in $|z|<1$, metrized by the norm $\|f\|=\sup|a_n|^{1/n}$, and let S be the subset of R consisting of 0 and the functions that are schlicht in $|z|<1$. Hornich [Math. Ann. **135** (1958), 189–191; MR **20** #5870] raised the question of the existence of isolated points of S, and the author answers this in the affirmative.

F. Bagemihl (Ann Arbor, Mich.)

9612:

Kocur, M. F. On a class of univalent functions of V. A. Zmorovič. Dopovidi Akad. Nauk Ukraïn. RSR **1959**, 1060–1063. (Ukrainian. Russian and English summaries)

The author considers two classes of functions regular for $|z|<1$. $C(\varphi)$ is the class of functions of the form (1) $P(z)=(1/2\pi)\int_{-\pi}^{\pi}(1-ze^{i(\varphi-\theta)})(1-ze^{-i\theta})^{-1}d\mu(\theta)$, where $d\mu$ is a distribution with total mass 2π on $(-\pi, \pi)$. $A(\hat{\varphi})$ is the class of functions $f(z)$, $f(z)=z+a_2z^2+\cdots$, with $f'(z)\in C(\varphi)$. Using (1), the author obtains a number of elementary inequalities for members of $C(\varphi)$ and $A(\hat{\varphi})$. For example, if $f(z)\in A(\hat{\varphi})$ then $|a_n|<2n^{-1}\cos(\varphi/2)$. Most of the results are generalizations of known ones for the case $\varphi=0$.

E. Reich (Aarhus)

9613:

Luke, Yudell L. The radius of univalence of the function $\exp z^2\int_0^z\exp(-t^2)dt$. Numer. Math. **3** (1961), 76–78.

Using the theory of Padé approximants [Y. L. Luke, J. Math. and Phys. **37** (1958), 110–127; MR **20** #5558] two sets of polynomials $\{\pi_n(y)\}$ are obtained for which $\lim y_{0n}=\rho$, where y_{0n} is the smallest zero of $\pi_n(y)$ and ρ the radius of univalence of the title [E. Kreyszig and J. Todd, Pacific J. Math. **9** (1959), 123–127; MR **21** #5759]. Convergence is rapid, y_{04} being correct to about 8 decimals. Similar results can be obtained in the case of the error function.

John Todd (Pasadena, Calif.)

9614:

Beesack, Paul R. Linear differential equations and convex mappings. Duke Math. J. **27** (1960), 483–495.

Let $f(z)=z^{-n}+a_1z^{-n+1}+\cdots$ $(n\geq 1)$ be regular for $0<|z|<1$, and denote by $\{f(z), z\}_n$ the expression $(f''/f')'-(n+1)^{-1}(f''/f')^2$ which, for $n=1$, reduces to the Schwarzian derivative. The author shows that if $\{f(z), z\}_n$ satisfies certain conditions, the function $f(z)$ will map $|z|=1$ onto a curve which is convex of order $-n$. These conditions are expressed in terms of the solutions of the real differential equation $y''+p(x)y=0$, where $p(x)$ is obtained from $\{f(z), z\}_n$ in a simple manner. If $n>1$ and $f(z)$ leaves out a value w_0, the same conditions will assure that $f(z)$ is n-valent and starlike of order $-n$ with respect to w_0.

Z. Nehari (Pittsburgh, Pa.)

9615:

Charzynski, Z.; Schiffer, M. A geometric proof of the Bieberbach conjecture for the fourth coefficient. Scripta Math. **25** (1960), 173–181.

The authors give a rather short proof of the Bieberbach conjecture that for univalent functions in the unit circle with Taylor series development $z+\sum_{n=2}^\infty a_nz^n$, the inequality $|a_4|\leq 4$ holds. This proof is modeled after a previous proof by Garabedian and Schiffer that $|a_3|\leq 3$ [Ann. of Math. (2) **61** (1955), 116–136; MR **16**, 579]. The proof uses the differential equation satisfied by the extremal function which is obtained by variational methods. It also makes use of an elementary estimate which says that the function which maximizes $|a_4|$ with a_4 real also satisfies $\mathrm{Re}(a_2)>1.6$ [Garabedian and Schiffer, J. Rational Mech. Anal. **4** (1955), 427–465; MR **17**, 24]. The proof is based upon geometric considerations of the trajectories of a differential equation of the form

$$\mathrm{Re}\int(D_1t+D_2t^2+D_3t^3)^{1/2}\frac{dt}{t}=\text{const.}$$

The estimate for $|a_4|$ is obtained by first maximizing the functional $\Phi=x_4+2y_2y_3-x_2y_2^2-2y_2^3-x_2$, where $a_n=x_n+iy_n$, and then using known estimates for certain combinations of coefficients.

G. Springer (Lawrence, Kans.)

9616:

Tumarkin, G. C.; Havinson, S. Ya. Classes of analytic functions on multiply connected domains. Issledovaniya po sovremennym problemam teorii funkciĭ kompleksnogo peremennogo, pp. 45–77. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960. (Russian)

In addition to the well-known classes A and H_p, two classes D and E_p (introduced by V. I. Smirnov in 1932) are considered. Chapter I deals with analytic functions in the unit-circle. Definitions of A, D, and H_p are given and their characterization by properties leading to equivalent definitions shown. In chapter II these definitions are extended to arbitrary (multiply-connected) domains G. A proof is given of a theorem stating necessary and sufficient conditions for a single-valued analytic $f(z)$ in G to be in $D(G)$. Relations between $A(G)$, $D(G)$, $H_p(G)$ and the classes A, D, H_p in the unit-circle are studied. Three decomposition theorems for $f(z)\in A(G)$, $D(G)$, $H_p(G)$ are proven.

Chapter III deals with functions in finitely-connected domains. The definition of E_p, given by V. I. Smirnov for the simply-connected case, is extended to the class $E_p(G)$.

Necessary and sufficient conditions for $f(z) \in E_p(G)$ are established.

Boundary values of functions belonging to $A(G)$, $D(G)$, $H_p(G)$ and $E_p(G)$ are investigated. Conditions are established for analytic functions in multiply-connected domains to be represented by Cauchy and Green formulas. Finally [see the authors, Izv. Akad Nauk SSSR Ser. Mat. **22** (1958), 379–386; MR **20** #6512] the class S of domains (Smirnov domains) is defined for n-connected domains G and a theorem stated, giving necessary and sufficient conditions for $G \in S$. *B. A. Amirà* (Jerusalem)

9617:

Ozawa, Mitsuru; Mizumoto, Hisao. On rings of analytic functions. Japan. J. Math. **29** (1959), 114–117.

Let $\mathrm{Log}^+(W)$ denote the ring of all single-valued analytic functions f on the Riemann surface W such that $\log^+|f|$ has a harmonic majorant on W. Theorem: If D_1 and D_2 are plane domains whose complements have positive capacity, and if ψ is a ring-isomorphism of $\mathrm{Log}^+(D_1)$ onto $\mathrm{Log}^+(D_2)$ which preserves constants, then there is a one-to-one conformal map φ of D_2 onto D_1 such that $f(\varphi(w)) = (\psi f)(w)$ for all $w \in D_2$ and $f \in \mathrm{Log}^+(D_1)$. This is analogous to known theorems concerning rings of bounded analytic functions. The hypothesis concerning positive capacity cannot be removed from the theorem. Also, it is shown that there exist two Riemann surfaces W_1 and W_2 which are not conformally equivalent, although the rings $\mathrm{Log}^+(W_1)$ and $\mathrm{Log}^+(W_2)$ are isomorphic. *W. Rudin* (Madison, Wis.)

9618:

Dunduženko, L. E.; Kas'yanyuk, S. A. Sulle funzioni C in un annello circolare. Bul. Inst. Politehn. Iaşi (N.S.) **5 (9)** (1959), nos. 1–2, 41–46. (Russian. Italian and Romanian summaries)

The authors consider the class of functions $\sum_{-\infty}^{\infty} a_k z^k$ which are analytic with positive real part in a circular annulus, and obtain exact estimates for the quantities $\mathrm{Im}\, f(z)$, $|f(z)|$, $|f'(z)|$, $\arg f(z)$, and $|a_k|$. The proofs are based on an integral representation for this class due to Zmorovič [Mat. Sb. (N.S.) **32 (74)** (1953), 633–652; MR **14**, 1075]. *A. J. Lohwater* (Houston, Tex.)

9619:

Yoshikawa, Hirosaku. On the minimum modulus of Blaschke product. Mem. Fac. Sci. Kyushu Univ. Ser. A **14** (1960), 111–121.

The author obtains criteria for limits of the minimum modulus of a Blaschke product in terms of its zeros. Let $B(z)$ be a Blaschke product having zeros at points $-x_n$ where $0 \leq x_n \nearrow 1$. Define $m(r) = \min_\theta |B(re^{i\theta})|$, $\sigma = \lim\sup_{r \to 1} m(r)$,

$$p_n = \Big(\sum_{k=1}^{n} (1-x_k)^{-1}\Big)\Big(\sum_{k=n+1}^{\infty} (1-x_k)\Big),$$

and $h_n = (1-x_{n+1})/(1-x_n)$. The author proves: (1) $\sigma = 1$ if and only if $\liminf p_n = 0$; (2) there exists a sequence $\{\xi_n\}$ such that $x_n \leq \xi_n \leq x_{n+1}$ and $\lim m(\xi_n) = 1$ if and only if $\lim h_n = 0$; (3) $\sigma = 0$ if and only if $\lim p_n = \infty$.

Defining $m_n = \max_{x_n \leq r \leq x_{n+1}} m(r)$ and $\tau = \liminf m_n$, then (4) $\tau = 0$ if and only if $\limsup h_n = 1$. Results on intermediate values for these limits are stated as consequences of the above theorems. Simple examples are given.

As observed by the author, M. Heins [Duke Math. J. **14** (1947), 179–215; MR **8**, 575] examined the role of the Blaschke product factor of a bounded analytic function in determining its minimum modulus. Certain of the above results extend to the general Blaschke product and thus contribute to the general minimum modulus problem. *G. Johnson* (Houston, Tex.)

9620:

Falgas, Maurice. Sur la définition des séries de base de polynômes. C. R. Acad. Sci. Paris **249** (1959), 2705–2707; **250** (1960), 43–45.

In the present papers the author generalises the definitions of basic series of polynomials given previously by J.-M. Whittaker [*Sur les séries de base de polynômes quelconques*, Gauthier-Villars, Paris, 1949; MR **11**, 344] and by R. C. Buck and R. P. Boas [*Expansion theorems for analytic functions*, mimeographed, Univ. of Wisconsin, Madison, Wis., 1954; cf. Buck, *Lectures on functions of a complex variable*, pp. 409–419; Univ. of Michigan Press, Ann Arbor, Mich., 1955; MR **17**, 140]. In this work $\mathscr{P}$ denotes the vector space of polynomials of one variable in the field of complex numbers, $\mathscr{V}$ is a vector space of functions and U is a linear application of $\mathscr{P}$ in $\mathscr{P}$, defined by a "generating series". The author defines the basic series of polynomials as follows.

Given a basic set (p_m) of polynomials, if the image of z^n by U is $\sum_{m=0}^{\infty} \pi_{n,m} p_m(z)$, $n = 0, 1, 2$, then the application $z^n \to \pi_{n,m}$ defines linear forms L_m over $\mathscr{P}$. Now suppose that the function $g \in \mathscr{V}$ and $\mathscr{F} \in U^{-1}(g)$; then $\mathscr{F}$ defines a basic series of g whenever $\lim_{f \in \mathscr{F}} L_m(f)$ exists for all m. If this limit is equal to A_m, then the basic series of g is $\sum_{m=0}^{\infty} A_m p_m(z)$.

The author then shows that this definition reduces to that of Whittaker when $\mathscr{V}$ is the vector space of functions regular at the origin. He also shows how the above definition reduces to that of Buck and Boas for basic series of functions defined by their Laurent expansion. *M. Nassif* (Assiut)

9621:

Bishop, Errett. Boundary measures of analytic differentials. Duke Math. J. **27** (1960), 331–340.

The author extends the results of his previous paper [same J. **25** (1958), 283–289; MR **20** #5880]. Let C be a compact plane set with the property that every boundary point of C is a boundary point of the unbounded component of the complement of C. Let U be the interior of C, and B the boundary of C. Say that $\{\gamma_i\}$ delimits U if each γ_i is a finite union of rectifiable Jordan curves in U, no two of which are in the same component of U, and such that every compact subset of U lies inside γ_i for sufficiently large i. If $d\omega$ is an analytic differential in U, put $\|d\omega\| = \inf \sup_i \int_{\gamma_i} |d\omega|$, where the inf is taken over all sequences $\{\gamma_i\}$ which delimit U. $H(U)$ is defined to be the space of norm-finite analytic differentials in U. Let $M(C)$ be the space of all measures on B which annihilate every function analytic in a neighborhood of C. If C is the closed unit disc, then $H(U)$ is the set of all $f(z)dz$ where f belongs to the Hardy class H^1, and $M(C)$ the set of measures on the circumference which annihilate polynomials. The author's main theorem, generalizing the F.

and M. Riesz theorem, establishes an isometry of $H(U)$ onto $M(C)$. A consequence is the famous theorem of Mergelyan: If C has connected complement, then every function continuous on C and analytic in U can be uniformly approximated by polynomials.

A. Browder (Providence, R.I.)

9622:

Epstein, Bernard; Minker, Jack. Extremal interpolatory problems in the unit disc. Proc. Amer. Math. Soc. **11** (1960), 777–784.

The interpolation problems considered are formulated with respect to the Hilbert spaces $L^2(\Delta)$ and $H_n^2(\Delta)$, the former consisting of all analytic functions square-integrable over the unit disc Δ and the latter being a weighted H^2 space. The authors apply Hilbert-space methods developed in their joint paper with the reviewer [Ann. Acad. Sci. Fenn. Ser. A I No. 250/10 (1958); MR **20** #2457] to find extremal interpolating functions. Although their results are not new (as they themselves admit), their use of Blaschke products to construct the projection operators which yield the extremal functions is of interest and should (it seems to the reviewer) in the weighted H^2 case be capable of yielding necessary and sufficient conditions for non-vacuous interpolation problems involving a denumerable infinity of interpolating conditions.

D. S. Greenstein (Evanston, Ill.)

9623:

Klingen, Helmut. Analytic automorphisms of bounded symmetric complex domains. Pacific J. Math. **10** (1960), 1327–1332.

One of Cartan's types of bounded symmetric domains is the domain E_n of $n \times n$ ($n \geq 2$) skew-symmetric complex matrices Z such that $I - ZZ' > 0$. It is well known that the identity component G_0 of the group G of holomorphic automorphisms of E_n consists of the mappings $Z \to (AZ+B)(-\bar{B}Z+\bar{A})^{-1}$ with $A^*A - B^*B = I$, $A^*B = -\overline{B^*A}$. In the present paper the author determines the full group G; for $n \neq 4$ it turns out that $G = G_0$; for $n = 4$, G has two connected components.

A. Korányi (Berkeley, Calif.)

9624:

Gagua, M. B. Some applications of multiple integrals of Cauchy type. Issledovaniya po sovremennym problemam teorii funkcii kompleksnogo peremennogo, pp. 345–351. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960. (Russian)

The paper deals with a problem in the theory of functions of several complex variables. Considering the case of two variables, let L be a simple curve in the plane and a complex-valued function $f(z, \zeta)$ defined on (L, L) ($z \in L$, $\zeta \in L$) continuous in z and in ζ. $\int_{(L,L)} f(z,\zeta)dzd\zeta$ is defined as the usual limit of a Riemann sum. Let L be a closed simple curve, the common boundary of the finite domain T^+ and the infinite T^-. Then

$$(*) \qquad \Phi(z,\zeta) = -\frac{1}{4\pi^2}\int_{(L,L)} \frac{f(t,\tau)\,dt\,d\tau}{(t-z)(\tau-\zeta)}$$

is analytic in the four cylindric domains (T^+, T^+), (T^+, T^-), (T^-, T^-), (T^-, T^+). The limiting boundary values of Φ on (L, L) are denoted by Φ^{++}, Φ^{+-}, Φ^{--}, Φ^{-+}.

A function $f(t, \tau)$ "satisfies the Hölder condition" on (L, L) if for every $(t, \tau) \in (L, L)$: $|f(t+\Delta t, \tau+\Delta\tau) - f(t, \tau)| \leq M(|\Delta t|^\alpha + |\Delta\tau|^\beta)$ with some positive constants M, α, β ($0 < \alpha \leq 1$, $0 < \beta \leq 1$).

A function $F(z, \zeta)$ is "piecewise holomorphic" if F is analytic in z and in ζ in the domains (T^+, T^+), (T^+, T^-), (T^-, T^-), (T^-, T^+), and if $\lim_{z,\zeta\to\infty} F(z,\zeta) = \lim_{\zeta\to\infty} F(z,\zeta) = \lim_{z\to\infty} F(z,\zeta) = 0$. Let $\bar{H}_{\alpha\beta}$ denote the class of piecewise holomorphic functions Φ for which Φ^{++}, Φ^{+-}, Φ^{--}, Φ^{-+} satisfy on (L, L) the Hölder condition with exponents α and β, and $H_{\alpha\beta}$ denote the class of arbitrary functions defined on (L, L) and satisfying the same condition there. The problem is to solve the equation

$$A(t,\tau)\Phi^{++} + B(t,\tau)\Phi^{+-} + C(t,\tau)\Phi^{--} + D(t,\tau)\Phi^{-+} = f(t,\tau)$$

with A, B, C, D and f given functions on (L, L) of the class $H_{\alpha\beta}$ and Φ^{++}, Φ^{+-}, Φ^{--}, Φ^{-+} boundary values of the required piecewise holomorphic function of class $H_{\alpha\beta}$. Solutions are given in different cases in terms of Cauchy type integrals (*).

B. A. Amirà (Jerusalem)

SPECIAL FUNCTIONS

See also 9575, 9756, 9777, 9818.

9625:

Boyd, A. V. Inequalities for Mills' ratio. Rep. Statist. Appl. Res. Un. Jap. Sci. Engrs. **6**, 44–46 (1959).

Generalizing previous work of Komatu [same Rep. **4** (1955), 69–70; MR **18**, 155] and the reviewer [ibid. **4** (1956), 110; MR **18**, 722], the author bounds

$$\exp(x^2/2)\int_x^\infty \exp(-t^2/2)\,dt$$

for $x > 0$ from above and below by expressions of the form $\alpha((x^2+\beta)^{1/2}+\gamma x)^{-1}$. For good approximations, $\alpha = \gamma + 1$ and $\beta = 2\alpha^2/\pi$. Then $\gamma = 2/(\pi-2)$ and $\gamma = \pi - 1$ give the sharpest upper and lower bounds respectively. The exact value of x_0 under (2) in the paper is incorrect, but this does not affect the result. For a generalization of Komatu's theorem to $\exp(x^p)\int_x^\infty \exp(-t^p)dt$, see Gautschi [J. Math. Phys. **38** (1959/60), 77–81; MR **21** #2067].

H. O. Pollak (Murray Hill, N.J.)

9626:

Philip, J. R. The function inverfc θ. Austral. J. Phys. **13** (1960), 13–20.

For the function inverfc defined as the inverse function of $2\pi^{-1/2}\int_x^\infty \exp(-\zeta^2)d\zeta$, the author derives a number of basic properties. These include a series for inverfc θ in powers of $1-\theta$, whose general term is unknown, approximations for small θ, an iteration formula for derivatives, indefinite integrals, and a small table of inverfc θ and of $B(\theta) = 2\pi^{-1/2}\exp(-\text{inverfc}^2\,\theta)$. The results are used by the author in his study of exact solutions of the concentration-dependent diffusion equation [#9627].

H. O. Pollak (Murray Hill, N.J.)

9627:

Philip, J. R. General method of exact solution of the concentration-dependent diffusion equation. Austral. J. Phys. **13** (1960), 1–12.

The author suggests a series of special functions $D(\theta)$ for which the equation

$$\frac{\partial t}{\partial \theta} = \frac{\partial}{\partial x}\left(D(\theta)\frac{\partial \theta}{\partial x}\right)$$

subject to the boundary conditions (1) $\theta=0$, $x>0$, $t=0$ and $\theta=1$, $x=0$, $t>0$ or (2) $\theta=0$, $x>0$ and $\theta=1$, $x<0$, $t=0$, $\int_0^1 x\,d\theta=0$, $t\geq 0$ can be solved explicitly. The proposed inverse method picks suitable functions θ and finds the corresponding D. Keeping $D(\theta)$ from 0 and ∞ as $\theta\to 0$ requires consideration of forms which involve the function inverse to erfc x. This function is studied in a companion paper [#9626].

H. O. Pollak (Murray Hill, N.J.)

9628:

Saaty, Thomas L. A note on the Fourier transform of Bessel functions. Atti Acad. Sci. Torino Cl. Sci. Fis. Mat. Nat. **93** (1958/59), 284–287. (Italian summary)

The author uses known formulas for integrals involving products of Bessel functions of the first kind in order to derive a formula for

$$i\int_{-\infty}^{\infty} e^{-ivt}J_\nu(ar)H_\nu^{(1)}(br)\,dr$$

if $\mathrm{Re}(\nu)>-\frac{1}{2}$, a, $b>0$. The result is 0 if $|t|<|a-b|$, $(ab)^{-1/2}P_{\nu-1/2}((a^2+b^2-t^2)/2ab)$ if $|a-b|<|t|<a+b$, and $(ab)^{-1/2}\cos\nu\pi Q_{\nu-1/2}((t^2-a^2-b^2)/2ab)$ if $|t|>a+b$.

H. O. Pollak (Murray Hill, N.J.)

9629:

Sharma, K. C. Infinite integrals involving products of Legendre functions. Proc. Glasgow Math. Assoc. **3**, 111–118 (1957).

An integral relation is derived between Laplace transforms and K-transforms. This relation is then used for the evaluation of two integrals containing products of Legendre functions in the integrand and of one integral containing the product of a Legendre function and a MacRobert function. Several special cases are considered.

D. J. Hofsommer (Amsterdam)

9630:

Saxena, R. K. Some integrals involving E-functions. Proc. Glasgow Math. Assoc. **4**, 178–185 (1960).

In this paper a number of integrals involving E-functions are evaluated. The results are exceedingly complicated and it is impossible to reproduce them here. The integrands contain products of E-functions, generalised hypergeometric functions, Bessel functions and exponential functions. In all cases the answers are expressed in E-functions.

Initially, the author derives the following formula: If $\psi(p)$ and $\phi(p)$ are the p-multiplied Laplace transforms of $f(t)$ and $t^{\alpha\alpha-1}f(t^\alpha)$ respectively, then

$$\phi(p) = (2\pi)^{-(\alpha+1)/2}\alpha^{\alpha\alpha-1/2}p^{1-\alpha\alpha}$$
$$\times\int_0^\infty \frac{1}{t}\sum_{i,-i}\frac{1}{i}E\left(a, a+\frac{1}{\alpha}, \cdots, a+\frac{\alpha-1}{\alpha} :: \frac{p^\alpha e^{i\alpha}}{\alpha^\alpha t}\right)\psi(t)\,dt$$

where $\sum_{i,-i}$ means that, in the expression following it, i is replaced by $-i$ and the two expressions are to be added.

J. L. Griffith (Kensington)

9631:

Engelis, G. K. Some families of polynomials related to Laguerre polynomials. Latvijas Valsts Univ. Zinātn. Raksti **28** (1959), no. 4, 99–110. (Russian. Latvian summary)

In the first section, the author considers a sequence of polynomials $y_n=y_n(a, b, c_k; x)$ such that the Laplace transform $z_n=\mathscr{L}y_n$ of y_n satisfies the recurrence relation

$$z_n(p) = z_{n-1}(p)(a+bp^{-1})+c_np^{-1}$$

(this generalises the relation satisfied by the Laplace transform of the Laguerre polynomial). Two particular cases, namely $a=0$, $b=1$ and $a=1$, $b=-1$ are studied, and many properties obtained for the corresponding families of polynomials.

In the second section the author studies a certain sequence of functions $H_k(x)$ such that their Laplace transforms are of the form $P(p)/(p+a)^{m+1}(p+b)^{n+1}$ where P is a polynomial of degree $\leq m+n+1$.

F. M. Arscott (Battersea)

9632:

Wright, E. M. A recursion formula for the coefficients in an asymptotic expansion. Proc. Glasgow Math. Assoc. **4**, 38–41 (1958).

The problem considered is that of determining an asymptotic expansion, for large x, of the generalized hypergeometric function $G(x)=\sum_{n=0}^\infty g(n)x^n$, where

$$g(t) = \prod_{r=1}^{p}\Gamma(t+\beta_r)\Big/\prod_{r=0}^{q}\Gamma(t+\gamma_r),$$

and $q\geq p\geq 0$.

From the known asymptotic expansion of the Γ-function one can obtain the resulting known asymptotic expansion of $G(x)$. Recurrence formulas for the coefficients in these expansions are also known. In this paper the author shows that $G(x)$ and its asymptotic expansion alike satisfy a simple differential equation and that from this a finite recurrence formula for these coefficients can be deduced. It is also shown how to calculate the coefficients c_m in an efficient way, depending on whether m is smaller than, about equal to, or larger than q.

T. E. Hull (Vancouver, B.C.)

9633:

Uvarov, V. B. A class of integral representations for hypergeometric functions. Vyčisl. Mat. **6** (1960), 17–33. (Russian)

The author considers the representation of functions φ in the form $\varphi(x)=\int_p^q F(z)\rho(z)/(z-x)^\lambda dz$ where $F(z)$ is a particular solution of a hypergeometric equation, and $\rho(z)$ is the coefficient of the highest derivative in that equation when it is in self-adjoint form. Here x is a real or complex number not lying on the contour joining p and q, except possibly at an end point for certain values of ζ.

It transpires that if F, p and q are chosen so as to satisfy a certain boundary condition, then φ is a solution of a different hypergeometric equation, whose parameters depend on ζ, and thus certain solutions of this new equation can be expressed as above. As a typical result,

the author shows that if $\mathrm{Re}\,\gamma>0$, and $\mathrm{Re}(\beta-\gamma)>-1$, then for $|x|>1$,

$$\int_0^1 F(-n,\beta+n;\gamma;z)z^{\gamma-1}(1-z)^{\beta-\gamma}/(x-z)^{\zeta}\,dz = (-1)^n\frac{\Gamma(\gamma)\Gamma(\beta-\gamma+n+1)}{\Gamma(\zeta)\Gamma(\beta+2n+1)}\Gamma(n+\zeta)x^{-(n+\zeta)} \times F(n+\zeta,n+\gamma;\beta+2n+1;x^{-1}).$$

The author also applies his method to confluent hypergeometric and Hermite functions, obtaining similar representations. *P. G. Rooney* (Toronto)

9634:

Rathie, C. B. Integrals involving E-functions. Proc. Glasgow Math. Assoc. **4**, 186–187 (1960).

By a formal substitution for certain of the Gamma functions in the definition of the E-functions and an interchange of order of integration, the author is able to express three infinite integrals in terms of an E-function.

The first integral involves two Whittaker functions and an E-function; the second, two Bessel K-functions and an E-function; the third two circular functions and an E-function.

The details of the constants in the expressions are too complicated to warrant reproduction here.

J. L. Griffith (Kensington)

9635:

Shukla, Harishanker. On certain relations between products of bilateral hypergeometric series. Proc. Glasgow Math. Assoc. **3**, 141–144 (1957).

In this paper, the author gives a generalization of the reviewer's theorem connecting basic bilateral hypergeometric series. The author's theorem involves the sum of products of these generalized series. He also gives the analogue of this result for ordinary generalized bilateral hypergeometric series, together with a further even more general relationship connecting the sum of products of these series. This last result does not seem to have any basic analogue. *L. J. Slater* (Cambridge, England)

9636:

Carlitz, L. Multiplication formulas for generalized Bernoulli and Euler polynomials. Duke Math. J. **27** (1960), 537–545.

The Bernoulli and Euler polynomials of order p are defined by

$$\left(\frac{t}{e^t-1}\right)^p e^{xt} = \sum_{m=0}^{\infty} B_m^{(p)}(x)\frac{t^m}{m!},$$

$$\left(\frac{2}{e^t+1}\right)^p e^{xt} = \sum_{m=0}^{\infty} E_m^{(p)}(x)\frac{t^m}{m!},$$

where p is an arbitrary positive integer. In the present paper the author introduces the polynomial $\phi_m^{(p)}(x,\zeta)$ defined by means of

$$\left(\frac{1-\zeta}{1-\zeta e^t}\right)^p e^{xt} = \sum_{m=0}^{\infty}\phi_m^{(p)}(x,\zeta)\frac{t^m}{m!},$$

where the parameter ζ is a primitive fth root of unity, $f>1$. Generalizing results of an earlier paper [Pacific J. Math. **9** (1959), 661–666; MR **21** #7317] he derives certain types of multiplication formulas for these polynomials. For example, he proves the following theorem. Let $\bar{B}_m^{(p)}(x)$ be defined by means of $\bar{B}_m^{(p)}(x)=B_m^{(p)}(x)$ $(0\le x<1)$, $\bar{B}_m^{(p)}(x+1)=\bar{B}_m^{(p)}(x)$. Let $k\ge 1$, $n\ge 1$, $p\ge 1$; let $m_1,\cdots,m_n\ge 0$; $a_1,\cdots,a_n$ be positive integers that are relatively prime in pairs; set $A=a_1a_2\cdots a_n$. Then we have

$$\sum_{r_j=0}^{k-1}\bar{B}_{m_1}^{(p)}\left(x_1+\frac{r_1+\cdots+r_p}{a_1k}\right)\cdots \bar{B}_{m_n}^{(p)}\left(x_n+\frac{r_1+\cdots+r_p}{a_nk}\right) = C\sum_{r_j=0}^{k-1}\bar{B}_{m_1}^{(p)}\left(a_1x_1+\frac{r_1+\cdots+r_p}{k}\right)\cdots \bar{B}_{m_n}^{(p)}\left(a_nx_n+\frac{r_1+\cdots+r_p}{k}\right),$$

where each summation is p-fold and $C=a_1^{p-m_1}a_2^{p-m_2}\cdots a_n^{p-m_n}$. The author also generalizes the Bernoulli and Euler polynomials in another direction and derives further extensions of his theorems.

A. L. Whiteman (Princeton, N.J.)

9637:

Eweida, M. T. A note on Legendre polynomials. Bull. Coll. Sci. Baghdad **4** (1959), 57–59. (Arabic summary)

In refining Turán's inequality, the author proves that the function

$$\varphi_n(x)=\frac{[P_n(x)]^2-P_{n-1}(x)P_{n+1}(x)}{1-x^2}$$

is strictly positive in the interval $-1\le x\le 1$ and has its minimum at $x=0$, and that $\varphi_n(\pm 1)=\frac{1}{2}$. Also $\int_{-1}^{1}\varphi_n(x)dx=2/(n+1)$. *G. Szegö* (Stanford, Calif.)

ORDINARY DIFFERENTIAL EQUATIONS

See also 9701, 9754, 9831, 9843, 9961, B10252, B10253, B10256, B10285, B10286, B10845.

9638:

Yosida, Kôsaku. ★Lectures on differential and integral equations. Pure and Applied Mathematics, Vol. X. Interscience Publishers, New York-London, 1960. ix+220 pp. $7.00.

We have here a modern introductory monograph on the theory of ordinary differential equations, together with those aspects of integral equations which are required for the former. There are five main chapters, whose headings are self explanatory. These are: 1. The initial value problem for ordinary differential equations; 2. The boundary value problem for linear differential equations; 3. Fredholm integral equations; 4. Volterra integral equations; and 5. The general expansion theorem (Weyl-Stone-Titchmarsh-Kodaira's theorem). The book closes with some comments on non-linear integral equations, and an appendix which contains several theorems of a more advanced nature from the theory of functions of a complex variable.

It is the author's intention to take us through the basic development of the initial value and boundary value problems (regular as well as singular) of ordinary differential equations, and a very readable account is given. The

book can be read with profit by a student who has a thorough grounding in the elements of the real and complex calculus minus Lebesgue integration. Of special value are the many examples which are given to illustrate the theory as well as to fulfill the author's desire to let the reader know why one proceeds in a particular direction or why a theory fails and modifications are necessary. There are a number of minor misprints.

A. E. Heins (Ann Arbor, Mich.)

9630:

Chalkley, Roger. On the second order homogeneous quadratic differential equation. Math. Ann. **141**, 87–98 (1960).

The author makes a detailed study of the differential equation (*) $(ay''+by'+cy)y''+(py'+qy)(ry'+sy)=0$. After disposing of the solutions for which either $y''=0$ or $py'+qy=0$, he sets $t=(ay''+by'+cy)/(py'+qy)=-(ry'+sy)/y''$ and replaces (*) by two equations of the form $Tt'+U=0$, $Vy'+Wy=0$, where T, U, V, W are polynomials in t of respective degrees at most 2, 4, 2, 2.

E. R. Kolchin (New York)

9640:

Bruwier, L. Contribution à l'étude d'une équation différentielle linéaire du second ordre. Mathesis **69** (1960), 7–30.

On considère l'équation différentielle (*) $y''+(au+b)y'+(Au^2+Bu+C)y=0$, où $u=u(x)$ est une fonction dérivable, et a, b, A, B, C des paramètres arbitraires. L'auteur cherche les conditions auxquelles doivent satisfaire la fonction u et les constantes $a, b, \cdots$ pour que l'équation (*) admette une intégrale particulière de la forme $y=\exp(p\int u\,dx+qx)$, p et q désignant des constantes. Il démontre qu'il est nécessaire, que la fonction u soit une fonction homographique de x ou de e^{cx}. {Le problème posé, se trouve en liaison étroite avec la notion de réductibilité, des équations différentielles linéaires, introduite par Frobenius. Il est traité en détail pour l'équation (*) par le reviewer [Fac. Philos. Univ. Skopje Sect. Sci. Nat. Annuaire **5** (1952), no. 2; MR **16**, 476].}

B. S. Popov (Skopje)

9641:

Abdelkader, Mostafa A. Solutions by quadrature of Riccati and second-order linear differential equations. Amer. Math. Monthly **66** (1959), 886–889.

The equation $y'=e+fy+gy^2$ can be solved by the substitution $y=a-v'/(gv)$, provided $a(x)$ satisfies another Riccati equation, conjugate to the given one. The conjugate of the conjugate is the original equation; but the conjugate of the equation for $1/a$ is a new equation. The author gets a broad class of equations solvable by quadrature, describes the solutions explicitly, and shows how his criterion produces solutions of certain higher-order equations.

R. M. Redheffer (Los Angeles, Calif.)

9642:

Bruwier, L. Sur une équation différentielle linéaire. Mathesis **68** (1959), 105–110.

L'auteur montre que l'équation différentielle

$$\sum_{r=0}^{n}(ax+b)^r(r!)^{-1}f^{(r)}(cx+d)y^{(r)}=0,$$

où a, b, c, d sont des constantes et $f(x)$ un polynome, se ramène par la substitution $y=z\cdot\exp\{-ca^{-1}x\}$, pour $a\neq 0$, à une équation d'Euler. Le cas $a=0$ conduit à une équation à coefficients constants.

C. Corduneanu (Iași)

9643:

Ciorănescu, N. Les linéarisations de l'équation de Riccati. Bull. Math. Soc. Sci. Math. Phys. R.P. Roumaine (N.S.) **2** (**50**) (1958), 127–131.

A necessary and sufficient condition for the Riccati differential equation (*) $y'+P(x)y^2+Q(x)y+R(x)=0$ to be "linearized" by a first order system (**) $u'=A(x)u+B(x)v$, $v'=C(x)u+D(x)v$, in the sense that $y(x)$ is a solution of (*) if and only if $y(x)=u(x)/v(x)$ with $u(x)$, $v(x)$ a solution of (**), is shown to be that (**) is of the form $u'=A(x)u-R(x)v$, $v'=P(x)u+[A(x)+Q(x)]v$, where $A(x)$ is "arbitrary". Determination of the general solution of (*) through an integration by parts of an associated linearizing system is illustrated in the case of the original Riccati equation $y'+ay^2-bx^\alpha=0$.

W. T. Reid (Iowa City, Iowa)

9644:

Sugai, Iwao. Riccati's nonlinear differential equation. Amer. Math. Monthly **67** (1960), 134–139.

The conventional transformation $y=z'/(Qz)$ applied to the generalized Riccati equation $y'+P(x)y+Q(x)y^2=R(x)$ reduces it to a second-order linear differential equation in z. The author suggests the "new" transformation $y=(Rf)/f'$ which also reduces the Riccati equation to a second order linear differential equation on f. The advantages of the new transformation are that it handles a certain class of higher degree first order nonhomogeneous equations and the resultant linear differential equation does not virtually increase the order by one.

K. S. Miller (New York)

9645:

Ghircoiașiu, N. Remarque sur un type d'équations différentielles. Inst. Politehn. Cluj. Lucrări Ști. **1959**, 69–71. (Romanian. Russian and French summaries)

L'auteur cherche la solution de l'équation

$$y'=a_1(x)y+\cdots+a_n(x)y^n$$

sous la forme $y=C(x)\exp(\int a_1(x)dx)$ et il écrit les conditions dans lesquelles l'équation qui détermine la fonction C est à variables séparées.

G. Marinescu (Bucharest)

9646:

Viswanatham, B. On the continuation of solutions of nonlinear differential equations. Bul. Inst. Politehn. Iași (N.S.) **5** (**9**) (1959), no. 3–4, 25–28. (Russian and Romanian summaries)

À l'aide des inégalités différentielles de la forme $|y'(x)|\leq\omega(x,|y(x)|)$ ou $|y'(x)|\geq\omega(x,|y(x)|)$, l'auteur établit six théorèmes concernant le prolongement des solutions de l'équation (*) $y'=f(x,y)$. Si les solutions de l'équation $y'=\omega(x,y)$ sont définies pour $x\geq 0$ et $|f(x,y)|\leq\omega(x,|y|)$, alors toute solution de l'équation (*) est définie pour $x\geq 0$. Si $|f(x,y)|\geq\omega(x,|y|)$ ou $1/|f(x,y)|\leq\omega(x,|y|)$ et $\omega(x,y)$ satisfait à des conditions convenables, toute solution de l'équation (*) devient infinie pour une valeur finie de x. Voir aussi R. Conti, Boll. Un. Mat. Ital. (3) **11** (1956), 510–514 [MR **18**, 736].

C. Corduneanu (Iași)

9647:

Bandič, I. Sur la méthode généralisée des transformations canoniques d'Euler. Acad. Roy. Belg. Bull. Cl. Sci. (5) **46** (1960), 336–344.

Using an extension of a method due to Euler, the author shows that the second-order linear homogeneous differential equation (*) $a_0(x)y''+a_1(x)y'+a_2(x)y=0$ is integrable by quadratures whenever the coefficients in (*) are of the form $a_0(x)=\alpha_0x^2+\alpha_1x+\alpha_2$, $a_1(x)=(\beta_0x+\beta_1)+2(\alpha_0x^2+\alpha_1x+\alpha_2)\Delta_1(p)$, $a_2(x)=n[\beta_0-(n+1)\alpha_0]+(\beta_0x+\beta_1)\Delta_1(p)+(\alpha_0x^2+\alpha_1x+\alpha_2)\Delta_2(p)$, where α_0, α_1, α_2, β_0, β_1 are constants, $\Delta_j(p)=p^{(j)}(x)/p(x)$ for an arbitrary function $p(x)$, and $n=1, 2, \cdots$. The paper concludes with a variety of applications of this general result, especially to equations of the form (*) with polynomial coefficients.

W. T. Reid (Iowa City, Iowa)

9648:

McCarty, G. S., Jr. Solutions to Riccati's problem as functions of initial values. J. Math. Mech. **9** (1960), 919–925.

The author studies the following differential system in the scalar functions $U=U(x, y, z)$, $\Lambda=\Lambda(x, y, z)$, $V=V(x, y, z)$, $W=W(x, y, z)$:

$$(*)\qquad \begin{aligned} U_y &= a+2bU+cU^2,\\ \Lambda_y &= (b+cU)\Lambda,\\ V_y &= \frac{b+cU+(a+bU)z}{1-zU}\,V-\frac{\Lambda(c+2bz+az^2)z}{1-zU}\,V^2,\\ W_y &= (c+2bz+az^2)V^2, \end{aligned}$$

satisfying the initial conditions $U(x, x, z)=z$, $\Lambda(x, x, z)=1$, $V(x, x, z)=1$, $W(x, x, z)=z$, where a, b and c are complex-valued functions of the real variable y. For solutions of (*) the author obtains functional equations and related differential equations satisfied by U_x, Λ_x, V_x, W_x, which are generalizations of those originally derived by Redheffer [J. Rational Mech. Anal. **5** (1956), 835–848; Math. Ann. **133** (1957), 235–250; MR **19**, 558, 655] for a system equivalent to (*) when the complex parameter z is equal to zero, together with additional properties and differential equations satisfied by U, Λ, V, W as functions of z. The discussion of the author utilizes some of the methods and results of the reviewer [J. Math. Mech. **8** (1959), 221–230; MR **22** #791] for the extension of the cited work of Redheffer to corresponding matrix differential systems.

W. T. Reid (Iowa City, Iowa)

9649:

Garnier, René. Sur des systèmes différentiels du second ordre dont l'intégrale générale est uniforme. C. R. Acad. Sci. Paris **249** (1959), 1982–1986.

The author looks for the systems of the form $dy/dx=P(y, z)$, $dz/dx=Q(y, z)$, where P and Q are homogeneous functions of the order $m\neq 1$, whose general solution is uniform.

M. Zlámal (Brno)

9650:

Kushul', M. A. On the quasiharmonic systems, close to systems with constant coefficients, in which pure imaginary roots of the fundamental equation have nonsimple elementary divisors. J. Appl. Math. Mech. **22** (1958), 721–740 (519–533 Prikl. Mat. Meh.).

Consider the system of linear differential equations (1) $dx/dt=[A+\mu F(t, \mu)]x$, $t\geq 0$, where A is a constant matrix, μ is a real parameter, $F(t, \mu)=F(t+w, \mu)$ is continuous in t and analytic in μ for $0\leq\mu\leq\mu_0$. The equation for the roots, ρ, of (1) $D(\rho, \mu)=0$, is an analytic function of μ, but the characteristic multipliers may have branch points of finite order at $\mu=0$. The order of a branch point depends very strongly upon the elementary divisors of A. The important problem in the applications is the determination of those characteristic multipliers, $\rho(\mu)$, of (1) such that $|\rho(0)|=1$, and the following discussion deals only with these. The characteristic multipliers at $\mu=0$ are determined by the eigenvalues of A. For the case where the elementary divisors of A are simple, the characteristic multipliers are generally analytic functions of μ, and many methods have been devised for their calculation [see, e.g., L. Cesari #9673; I. G. Malkin, *Teoriya ustoičivosti dviženiya*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1952; MR **15**, 873]. The present paper treats the case where the elementary divisors of A are not simple. The order of the branch point at $\mu=0$ of a characteristic multiplier is discussed and explicit formulas are given when the order is 1, 2 or 3. In particular, if $\rho(0)=\exp(i\sigma w)$ is a branch point of order three, then there exists a solution of (1) which is unbounded. On the other hand, if this branch point is of order two, then the boundedness of the solutions depends upon the sign of a certain constant.

J. K. Hale (Baltimore, Md.)

9651:

Valat, Jean. Influence de la dissymétrie du créneau sur les diagrammes de stabilité d'une équation de Hill à coefficient périodique crénelé. C. R. Acad. Sci. Paris **251** (1960), 2274–2276.

The author considers a linear homogeneous differential equation of the second order, the coefficient of the first order term being constant (attenuation), whilst the coefficient of the dependent variable is periodic, with abrupt changes between two constant values (Hill's differential equation). The general solution of this equation is well known, as is the expression for its characteristic exponent. The author refers to the special case that the time intervals, during which the second coefficient mentioned above assumes each of its two different values, are unequal, and gives diagrams of the frontiers of the zones, in a plane constituted by two significant parameters of the equation, in which the solutions of the equation are stable and unstable respectively. These diagrams are compared with those obtained in the case of equal time intervals (see above).

M. J. O. Strutt (Zürich)

9652:

Iséki, Kiyoshi. A remark on monotone solutions of differential equations. Proc. Japan Acad. **35** (1959), 370–371.

Demidovič [Uspehi Mat. Nauk **12** (1957), no. 2 (74), 143–146; MR **19**, 856] has proved that if the n-vector equation (1) $\dot{x}=P(t)x$, with $P(t)=(p_{ij}(t))$ real and continuous on $[a, \infty)$ and $P(t)$ integrable on $[a, \infty)$, (i.e., $\lim_{t\to\infty}\int^t p_{ij}$ exists (finite)), has a monotone solution $x(t)$ (each component monotone for large t), then $x(t)$ is bounded and, hence, has a limit $x(\infty)<\infty$. He also observed that the nonhomogeneous equation (2) $\dot{x}=P(t)x+$

$Q(t)$ can be put into the form (1), which yields immediately the corresponding result for monotone solutions of (2), where both matrices P and Q are integrable in the above sense.

The author applies Demidovič's method directly to (2) but only for the scalar case ($n=1$) and for $P(t)$ and $Q(t)$ absolutely integrable. Demidovič attributes the absolutely integrable case of (1) to Nemyckiĭ and Stepanov, *Kačestvennaya teoriya differencial'nyh uravnenii*, 2nd Russian edition, 1949 [for the review of 1st edition, OGIZ, Moscow, 1947, see MR **10**, 612].

J. H. Barrett (Salt Lake City, Utah)

9653:

Guggenheimer, Heinrich W. On monotone solutions of differential equations. Proc. Japan Acad. **35** (1959), 538–540.

Noting the above paper [#9652] the author poses the question of boundedness as $t\to\infty$ of monotone increasing solutions of the scalar equation (1) $dx/dt=\sum_{m=0}^{n}p_m(t)x^m$ and shows that if $\int^\infty p_n=+\infty$, $p_m\geqq 0$, $m=0, 1, \cdots, n$, for $t\geqq T$, then any monotone increasing solution of (1) is either bounded by zero or $\lim_{t\to\infty}(x(t)/\int^t p_n)=\infty$. Furthermore, if $|\int^\infty p_n|=+\infty$ and $\int^\infty |p_m|<\infty$, $m=0, 1, \cdots, n-1$, then any such solution is bounded by zero. He gives some simple motivating examples and remarks that his methods can be readily extended to analogous vector equations.

J. H. Barrett (Salt Lake City, Utah)

9654:

Dunn, Eldon L.; Stein, F. Max. Families of Sturm-Liouville systems. SIAM Rev. **3** (1961), 54–65.

The authors consider the boundary value problem

$$\begin{aligned} Py''+Sy'+(q+\lambda r)y &= 0,\\ (*)\qquad Ay(a)+By'(a) &= 0,\\ Cy(b)+Dy'(b) &= 0, \end{aligned}$$

where p, q, r, and s are real-analytic functions in the interval $[a, b]$, and $p>0$ in the interior of $[a, b]$; the authors refer to (*) as a Sturm-Liouville system, and they assume that when it is singular, its spectrum is discrete.

The problem the authors consider is that of determining under what circumstances all the boundary value problems that can be obtained from (*) by differentiating or integrating the differential equation are Sturm-Liouville systems, and they obtain necessary and sufficient conditions for this. They also investigate some of the properties of the eigenfunctions of the derived problems.

P. G. Rooney (Toronto)

9655:

Atkinson, F. V. Asymptotic formulae for linear oscillations. Proc. Glasgow Math. Assoc. **3**, 105–111 (1957).

The following theorem is proved. (*) Let $t\geqq 0$, $0<f(t)\in C^1$, $g(t)\in C^0$, $f'f^{-3/2}=h(t)+h_1(t)$, where $\int^\infty |gf^{-1/2}|dt<\infty$, $h_1\in C^0$ satisfies $\int^\infty |h_1 f^{1/2}|dt<\infty$, $\int_0^\infty |dh|<\infty$ and $|h(\infty)|<4$. Then every non-trivial solution $x=x(t)$ of (1) $x''+[f(t)+g(t)]x=0$ is of the asymptotic form $x=cf^{-1/4}(t)\times\cos\{\int_0^t f^{1/2}(1-h^2/16)^{1/2}dt+d+o(1)\}$ as $t\to\infty$ with c, d constants. If the independent variable $u=\int_0^t f^{1/2}dt$ is introduced, (1) becomes a linear system of differential equations for the binary vector $y=(x, dx/du)$ of the form $dy/du=(A+B(u)+C(u))y$, where A is a constant matrix, $\int^\infty |dB_{jk}|<\infty$, $\int^\infty |C_{jk}|du<\infty$ and the eigenvalues of $A+B$ are $-h/4\pm i(1-h^2/16)^{1/2}$. The result follows by an application of a result of Levinson [Duke Math. J. **15** (1948), 111–126; MR **9**, 509]. It is shown that (*) contains related results due to a number of authors.

P. Hartman (Baltimore, Md.)

9656:

Rosenfeld, Norman S. The eigenvalues of a class of singular differential operators. Comm. Pure Appl. Math. **13** (1960), 395–405.

Let $q(t)$ be continuous, increasing and $q(+\infty)=0$. Let $t=\varphi(\lambda)$ be the function inverse to $\lambda=q(t)$; $N(\lambda)$ the number of zeros of some solution of $x''+(\lambda-q)x=0$ for $t\geqq 0$; and $I(\lambda)$ the integral of $(\lambda-q(t))^{1/2}$ over $[0, \varphi(\lambda)]$ for $q(0)<\lambda<0$. The author gives conditions sufficient for $\pi N(\lambda)\sim I(\lambda)$, $\pi N(\lambda)=I(\lambda)+O(1)$ or $C_1 I(\lambda)\leqq N(\lambda)\leqq C_2 I(\lambda)$ as $\lambda\to-0$. His proofs are obtained by modifying the arguments of Atkinson [Univ. Nac. Tucumán. Revista A **8** (1951), 71–87; MR **14**, 50] and Hartman [J. London Math. Soc. **27** (1952), 492–496; MR **14**, 278] who deal with the case of $q(t)$ increasing to ∞ as $t\to\infty$. {Incidentally, the author's conditions $q''\leqq 0$ and $q''=o(q'^{4/3})$ as $t\to\infty$ in theorem 1, which is concerned with $\pi N(\lambda)\sim I(\lambda)$, can be relaxed to $q'(t)^{3}\to\infty$ as $t\to\infty$ and a proof obtained for the resulting theorem by reading "$q\to 0$, $t\to\infty$" and "as $\lambda\to 0$" instead of "$q\to\infty$, $t\to\infty$" and "as $\lambda\to\infty$", respectively, in Hartman [Amer. J. Math. **73** (1951), 657–662; MR **13**, 463].}

P. Hartman (Baltimore, Md.)

9657:

Wend, David V. V. On the zeros of solutions of some linear complex differential equations. Pacific J. Math. **10** (1960), 713–722.

In the first three sections of this paper, the author studies the oscillation of solutions of the differential equation $[p(x)y']'+f(x)y=0$, where $p(x)$ and $f(x)$ are complex-valued functions on the real axis. He shows that if $p(x)$ is subject to certain restrictions, some of the results known for $p(x)\equiv 1$ extend to this more general case. In sections 4 and 5, it is pointed out that estimates for the distribution of zeros of the differential equation $y^{(k)}+f(x)y=0$ ($k\geqq 3$) can be obtained if the Green's function of $y^{(k)}=0$ with k given zeros is known explicitly. This is illustrated by the case $k=3$, in which the Green's function in question can be constructed without too much trouble.

Z. Nehari (Pittsburgh, Pa.)

9658:

Perov, A. I. A two-point boundary value problem. Dokl. Akad. Nauk SSSR **122** (1958), 982–985. (Russian)

Let f be continuous on $[a, b]\times R\times R$ to R and consider the non-linear boundary problem (*) $y''=f(x, y, y')$, $y(a)=y(b)=0$. The author studies the existence, uniqueness and non-uniqueness of solutions of (*). The technique is to compare (signum y) $f(x, y, y')$ with continuous functions of the form $\alpha(x, y, y')+\beta(x)$ which are positively homogeneous in the variables (y, y') and satisfy a Lipschitz condition with respect to these variables. As a very special example, suppose that f does not depend on y' and that $b-a=\pi$; then if the values of $f(x, y)/y$ for $|y|$ sufficiently large lie strictly interior to some strip $[-(k+1)^2, -k^2]$ for some integer k, (*) has at least one solution.

R. G. Bartle (Urbana, Ill.)

9659:

Klokov, Yu. A. A limiting boundary problem for a system of second-order ordinary differential equations. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **1959**, no. 5, 197–204. (Russian)

The author considers the autonomous system (1) $\ddot{x} = f(x, \dot{x})$, where the vector-function $f(x, y)$ is supposed Lipschitzian in any bounded domain and, for some continuous non-negative function $c(r)$, satisfies the inequality $|f(x, y)| \leqq c(r)(1+y^2)$, with $r = |x_1| + \cdots + |x_n|$. He proves that under these assumptions, for any solution $x(t)$ of (1) the condition $x(\infty) = b$ (a constant vector) implies $\dot{x}(\infty) = 0$. This theorem is used to prove that the condition $f(b, 0) = 0$ is necessary for the existence of the solution $x(t)$ of (1) for which $x(0) = a$ and $x(\infty) = b$.

As a special case of (1) the author considers the system $\ddot{x} = A\dot{x} + Bx + f(x, \dot{x})$ (A and B are constant matrices) and gives sufficient conditions (too complicated to list here) for the existence and uniqueness of a solution $x(t)$ satisfying the boundary conditions $x(0) = a$, $x(\infty) = 0$ for sufficiently small vectors a.

Z. Opial (Kraków)

9660:

Uno, Toshio; Hong, Imsik. Some consideration of asymptotic distribution of eigenvalues for the equation $d^2u/dx^2 + \lambda\rho(x)u = 0$. Japan. J. Math. **29** (1959), 152–164.

Following the usual construction of the Cantor function, remove (1/3, 2/3) from [0, 1], let E_{11}, E_{12} be the remaining intervals; remove the interior middle third from E_{11}, E_{12} and let $E_{21}, \cdots, E_{24}$ be the remaining intervals, etc. Let E_{jk} ($k = 1, \cdots, 2^j$) be the remaining intervals after j steps; $E_j = \bigcup_k E_{jk}$ and $E = \bigcap_j E_j$. Let $\sigma_j(x)$ be the (absolutely) continuous, piecewise linear, monotone function, which is constant on intervals complementary to E_j and has slope $(3/2)^j$ on E_{jk}; $\sigma_j(0) = 0$ and $\sigma_j(1) = 1$. Finally, $\sigma_\infty(x) = \lim \sigma_j(x)$, $j \to \infty$. Let $G(x, y)$ be the Green function for $u''(x) = 0$, $u(0) = u(1) = 1$. For a fixed j, let $0 < \lambda_1^j < \lambda_2^j < \cdots$ be the eigenvalues of

$$(1_j) \qquad u(x) = \lambda \int_0^1 G(x, y)u(y)d\sigma_j(y)$$

and $\varphi_1^j(x)$, $\varphi_2^j(x)$, $\cdots$ the corresponding eigenfunctions normalized by (2_j) $\int_0^1 |\varphi|^2 d\sigma_j = 1$. It is shown that $\lambda_n = \lim \lambda_n^j$, $j \to \infty$, exists for $n = 1, 2, \cdots$; that $\varphi_n(x) = \lim \varphi_n^j(x)$, $j \to \infty$, exists uniformly for $0 \leqq x \leqq 1$; that λ_n is an eigenvalue of (1_∞) and φ_n is a corresponding eigenfunction normalized by (2_∞); that the only eigenvalues of (1_∞) are the λ_n and that these are simple; finally, that if $A(\lambda)$ is the number of $\lambda_n \leqq \lambda$, then $A(\lambda)\ \lambda^{-\alpha/(1+\alpha)}$ and its reciprocal are bounded as $\lambda \to \infty$, where $\alpha = \log 2/\log 3$ is the Hausdorff dimension of the Cantor set E.

P. Hartman (Baltimore, Md.)

9661:

Morrison, D. D. The deflation of eigenvalue problems with second-order ordinary differential operators. J. Soc. Indust. Appl. Math. **9** (1961), 28–30.

Let the eigenvalues, in an arbitrary order, be λ_1, λ_2, $\cdots$. To "deflate" the problem is to set up a sequence of problems of which the kth has just λ_k, λ_{k+1}, $\cdots$ for eigenvalues. The author shows that a Sturm-Liouville problem (*) $L(y) + \lambda y = 0$, $y(0) = y(1) = 0$ may be deflated by a sequence of problems of the form $L(y) + \lambda y = \sum_{i=1}^{k-1} a_i y_i$, $y(0) = y(1) = 0$, $y'(0) = 0$, in which y_i is the eigenfunction of (*) corresponding to λ_i and a_i is a non-zero constant. A later paper is promised, reporting numerical experience with this method.

T. M. Cherry (Melbourne)

9662:

Титчмарш, Э. Ч. [Titchmarsh, E. C.]. ★Разложения по собственным функциям, связанные с дифференциальными уравнениями второго порядка. Часть I [Eigenfunction expansions associated with second-order differential equations. Part I]. Translated from the English by V. B. Lidskiĭ; edited by B. M. Levitan. Izdat. Inostr. Lit., Moscow, 1960. 278 pp. 12.80 r.

The English original [Clarendon, Oxford, 1946] was reviewed in MR 8, 458. The present version comprises a translation of the original, together with an introduction and five appendices by B. M. Levitan, and a second bibliography. In translating the original the opportunity was taken of correcting a number of misprints and mistakes; this is particularly the case in sections 4.1 and 5.8, the latter being practically rewritten. Also, the example originally in section 4.14 has been shifted into section 5.8.

In his introduction, Levitan states that the purposes of the appendices are to simplify some of the results, modernize others, particularly those in chapters 6, 9, and 10, and to include certain other topics. Appendices 1, 2, 4, and 5 are entitled respectively: proof of the expansion theorem; some remarks on the problem of the nature of the spectrum; transform operators and inverse problems of spectral analysis; Tauberian theorems for Fourier integrals. Appendix 3 is concerned with the asymptotic behaviour of the spectral kernel, and the equi-convergence of an expansion deduced in appendix 2 with the Fourier cosine expansion. The second bibliography contains nine titles.

P. G. Rooney (Toronto)

9663:

Hayashi, C.; Shibayama, H.; Nishikawa, Y. Frequency entrainment in a self-oscillatory system with external force. IRE Trans. **CT-7** (1960), 413–422.

From author's introduction: "The phenomenon of frequency entrainment occurs when a periodic force is applied to a system whose free oscillation is of the self-excited type. A typical and important case is the system governed by van der Pol's equation with an additional term for periodic excitation. The frequency of the self-excited oscillation falls in synchronism with the driving frequency, provided these two frequencies are not far different. If their difference is large enough, one may expect the occurrence of a beat oscillation. However, the entrainment of frequency still takes place when the ratio between the natural frequency of the self-excited oscillation and the driving frequency is in the neighborhood of an integer (different from unity) or a fraction. Under this condition, the natural frequency of the system is entrained by a frequency which is an integral multiple or submultiple of the driving frequency. In this paper, special attention is directed toward the study of periodic oscillations as caused by frequency entrainment, and the types of resulting oscillation which are dependent on the amplitude and frequency of the external force are investigated for a particular set of the system parameters."

H. D. Block (Ithaca, N.Y.)

9664:

Mikolajska, Z. Sur l'existence de solutions périodiques de l'équation différentielle du second ordre dépendant d'un paramètre. Ann. Polon. Math. **6** (1959/60), 51–68.

By using Schauder's fixed-point theorem (the author proves fifteen lemmas to verify the hypotheses) and employing a method of A. Bielecki [Bull. Acad. Polon. Sci. Cl. III **4** (1956), 265–268; MR **18**, 494] the author gives sufficient conditions for the existence of at least one periodic solution $\psi(t, \lambda)$ of period T in t, for $|\lambda|$ sufficiently small, of the scalar equation (1) $x''=g(t, x, x', \lambda)$, where g has period T in t and λ is a parameter, assuming that the unperturbed equation (2) $y''=g(t, y, y', 0)$ also has a periodic solution $\psi_0(t)$ of period T. The solution $\psi(t, \lambda)$ is continuous in (t, λ) ($-\infty<t<\infty$, $|\lambda|$ sufficiently small) with $\psi(t, 0)=\psi_0(t)$. The conditions are (i) g, g_x, $g_{x'}$, g_λ are continuous in the whole space R_4 of variables (t, x, y, λ), (ii) $|g_x(t, \psi_0(t), \psi_0'(t), 0)|$, $|g_{x'}(t, \psi_0(t), \psi_0'(t), 0)|$, $|g_\lambda(t, \psi_0(t), \psi_0'(t), 0)| < M$ ($-\infty<t<\infty$, M a constant >0), (iii) $\left|\int_0^T g_x(t, \psi_0(t), \psi_0'(t), 0)dt\right| > m$ (m a constant >0), (iv) $m > (MT)^2(3T+7/2)$. Examples of a number of second-order equations are given. The theorem extends results of A. M. Kac [Prikl. Mat. Meh. **19** (1955), 13–32; MR **16**, 822] for $\ddot{x}+F(x)=\varepsilon f(t, x, \dot{x}, \varepsilon)$, where f has period T in t, f, F are analytic and $\ddot{y}+F(y)=0$ has a periodic solution $x_0(t)$ of period T. The examples given could also be treated by the method of Cesari [#9673], since the equations considered are actually "weakly nonlinear".

J. A. Nohel (Atlanta, Ga.)

9665:

Pliss, V. A. The existence of periodic solutions of certain non-linear systems. Dokl. Akad. Nauk SSSR **137** (1961), 1060–1062 (Russian); translated as Soviet Math. Dokl. **2**, 416–418.

Consider the system of differential equations (1) $dx/dt=Ax+F(x, t)$ where x and $F(x, t)$ are real n-dimensional vectors and A is a real constant quadratic matrix of order n. The function $F(x, t)$ is continuous, periodic in t with period 2π and satisfies the uniqueness condition of solutions for all x, t.

The author proves the following theorem. If the matrix A does not have characteristic roots of the form ki (k being a natural number or zero and i the imaginary unit) and if for sufficiently large $\|x\|=\sum_{\nu=1}^{n}|x_\nu|$ the function $F(x, t)$ satisfies the inequality $\|F(x, t)\|<L\|x\|$, where the positive constant L is sufficiently small, then system (1) has at least one periodic solution of period 2π [cf. also M. A. Krasnosel'skiĭ, Dokl. Akad. Nauk SSSR **111** (1956), 283–286; **123** (1958), 235–238; MR **19**, 144; **21** #3621].

E. Leimanis (Vancouver, B.C.)

9666:

Valat, Jean. Sur les courbes et mouvements de Lissajous non linéaires. C. R. Acad. Sci. Paris **251** (1960), 630–632.

En utilisant les résultats d'une Note antérieure [mêmes C. R. **247** (1958), 1961–1964; MR **20** #5565], l'auteur étudie le système $x''+\alpha x+\gamma x^3=0$, $y''+\beta y+\delta y^3=0$, où α, β, γ, δ sont des constantes. Sous certaines conditions, le système admet des mouvements bornés qui se divisent, comme ceux de Lissajous, en deux classes: les mouvements périodiques algébriques et les mouvements non algébriques dont les trajectoires sont quasi denses dans un domaine borné. Les équations paramétriques du mouvement s'obtiennent à l'aide des fonctions elliptiques.

C. Corduneanu (Iași)

9667:

Littlewood, J. E. On the number of stable periods of a differential equation of the van der Pol type. IRE Trans. **CT-7** (1960), 535–542.

The concern is with periodic solutions of $\ddot{y}+kf(y)\dot{y}+y=bkp(t)$ where $p(t+2\pi)=p(t)$; $f(y)$ is odd, and negative for $0<y<1$, positive for $y>1$; and k is a large positive constant. When $f(y)=y^2-1$, $p(t)=\cos t$ and b is suitably chosen, the author [Acta Math. **97** (1957), 276–308; MR **19**, 548] has shown that there are stable sub-harmonics whose periods are two different odd multiples of 2π. In the present paper he shows, by taking $p(t)$ to have nearly (but not exactly) a sub-multiple of 2π as period, that there can be at least three stable sub-harmonics of different periods. The argument is sophisticated and is given in outline only. There are a few obvious misprints.

T. M. Cherry (Melbourne)

9668:

Putnam, C. R. Unilateral stability and almost periodicity. J. Math. Mech. **9** (1960), 915–917.

Let $x'=f(x)$ be an incompressible (div $f=0$), conservative system of differential equations on an open invariant x-set in E_n. A solution $x=x(t)$ is called stable in the future (or past) if, for every $\varepsilon>0$, there is a $\delta_\varepsilon>0$ such that $|y(0)-x(0)|<\delta_\varepsilon$ implies that the corresponding solution $x=y(t)$ satisfies $|y(t)-x(t)|<\varepsilon$ for $t\geq 0$ (or $t\leq 0$). It is shown that if a solution $x=x(t)$ is bounded and stable in the future, then it is also stable in the past and is uniformly (Bohr) almost periodic. The proof depends on the recurrence theorem of Poincaré and an improvement of it due to Khintchine.

P. Hartman (Baltimore, Md.)

9669:

Bhatia, N. P. Kriterien für die exponentielle Stabilität der Nullösung eines Systems linearer Differentialgleichungen mit veränderlichen Koeffizienten. Z. Angew. Math. Mech. **41** (1961), 134–136.

Consider the system (1) $x'=A(t)x$ where $A(t)$ is an $n\times n$ matrix whose elements are continuous functions of t for $t\geq 0$. It is known that if the eigenvalues of $A(t)$ have negative real parts bounded away from zero for all $t\geq 0$, then the solution $x=0$ of (1) is not necessarily stable. The author states some additional conditions on $A(t)$ which he claims will insure the asymptotic stability of $x=0$. The proof is not given but is to appear in Mathematischen Nachrichten. An example is also given.

J. K. Hale (Baltimore, Md.)

9670:

Hale, J. K.; Stokes, A. P. Behavior of solutions near integral manifolds. Arch. Rational Mech. Anal. **6**, 133–170 (1960).

Lo scopo essenziale di questo lavoro è quello di estendere il concetto di stabilità orbitale introducendo la nozione di varietà integrale asintoticamente stabile.

Si considerino i due sistemi (1) $dx/dt=X(x)$; (2) $dx/dt=X(x)+X^*(t, x)$ dove x, X, X^* sono n-vettori reali ed X, X^* soddisfano condizioni che assicurano l'unicità delle rispettive soluzioni.

Ammettiamo che (1) possieda una famiglia di soluzioni $x=x^0[\omega_1(b^1)t+\varphi_1, \cdots, \omega_m(b^m)t+\varphi_m, b^1, \cdots, b^m]$ dove b^j è un k_j-vettore appartenente ad un insieme U_j, aperto o costituito da un sol punto, di un R^{k_j} euclideo, $\omega_j(b^j)$ è una data funzione scalare di b^j, φ_j è uno scalare, ed esistano m numeri reali p_j tali che $x^0(\theta_1, \cdots, \theta_j+p_j, \cdots, \theta_m, b^1, \cdots, b^m)=x^0(\theta_1, \cdots, \theta_m, b^1, \cdots, b^m)$ per $b^j \in U_j$, $-\infty < \theta_j < \infty$, $j=1, \cdots, m$. Posto $b=(b^1, \cdots, b^m)$, $U=U_1 \times \cdots \times U^m$ si definisca, per ogni fissato $b \in U$, l'insieme C_b come l'insieme dei punti (x, t) tali che $x=x^0(\theta_1, \cdots, \theta_m, b)$, $-\infty < \theta_j < \infty$, $j=1, \cdots, m$; $-\infty < t < \infty$ ed infine si definisca la varietà $\mathfrak{M}=\bigcup_{b \in U} C_b$. Tale varietà è invariante rispetto ad (1), ma non, in generale, rispetto a (2); $\mathfrak{M}$ può ridursi, in particolare ad un cilindro dello spazio (x, t) generato per traslazione lungo l'asse t da una soluzione periodica di (1) e può anche degenerare in una retta $x=x^0$ parallela all'asse t stesso.

Se $U^* \subset U$ si ponga $\mathfrak{M}^*=\bigcup_{b \in U^*} C_b$ e se con $N_\sigma(\mathfrak{M}^*)$ si indica un σ-intorno di $\mathfrak{M}^*$ nello spazio (x, t) si ponga inoltre $\mathfrak{R}_\sigma(\mathfrak{M}^*)=[N_\sigma(\mathfrak{M}^*) \cap (R^{n-1}-\mathfrak{M})] \cup \mathfrak{M}^*$. Si possono allora porre le seguenti definizioni: (a) se assegnati $\varepsilon > 0$, $U^* \subset U$, U^* chiuso, esistono $\sigma > 0$, $T \geq 0$ tali che ogni soluzione $x(t)$ del sistema perturbato (2) per la quale è $(x(T), T) \in \mathfrak{R}_\sigma(\mathfrak{M}^*)$ soddisfa la relazione $(x(t), t) \in N_\varepsilon(\mathfrak{M}^*)$ per $t \geq T$ ed inoltre la distanza $d[(x(t), t), \mathfrak{M}] \to 0$ per $t \to \infty$, allora si dice che la varietà $\mathfrak{M}$ è asintoticamente stabile rispetto al sistema (2); (b) se inoltre per ogni soluzione $x(t)$ di (2) tale che $(x(T), T) \in \mathfrak{R}_\sigma(\mathfrak{M}^*)$ esiste un $b_0 \in U$ (dipendente da $x(T)$) tale che la distanza $d[(x(t), t), C_{b_0}] \to 0$ per $t \to \infty$ si dice che la varietà $\mathfrak{M}$ è asintoticamente stabile rispetto al sistema (2) con ampiezza asintotica; (c) se $\mathfrak{M}$ è asintoticamente stabile rispetto a (2) e per ogni soluzione $x(t)$ di (2) tale che $(x(T), T) \in \mathfrak{R}_\sigma(\mathfrak{M}^*)$ esistono $b_0 \in U$, φ_{j0} reale, $j=1, \cdots, m$, tali che $d[(x(t), t), (y(t), t)] \to 0$ per $t \to \infty$, dove $y(t)=x^0(\omega_1(b_0^1)t+\varphi_{10}, \cdots, \omega_m(b_0^m)t+\varphi_{m0}, b_0)$ allora si dice che la varietà $\mathfrak{M}$ è asintoticamente stabile rispetto a (2) con ampiezza e fase asintotica.

I risultati più semplici riguardano il caso $m=1$ e si possono riassumere come segue:

(α) (Th. V.1) Il sistema (1) ammetta una famiglia (3) $x=x^0(\omega(b)t+\varphi, b)$ dipendente da $k+1$ parametri $b=(b_1, \cdots, b_k)$, φ, di soluzioni periodiche, $x^0(\theta+p, b)=x^0(\theta, b)$. Supposto $X \in C^2$, $x^0(\theta, b) \in C^3(\theta)$, $x^0(\theta, b) \in C^2(b)$, $\omega(b) > 0$, $\omega(b) \in C^2$, $b \in U$, U aperto di R^k, allora la varietà $\mathfrak{M}$ definita da (3) è asintoticamente stabile rispetto a (1) con ampiezza e fase asintotiche se $n-(k+1)$ esponenti caratteristici delle equazioni alle variazioni di (3) hanno parte reale negativa per ogni $b \in U$ ed inoltre il rango della matrice $[x_\theta(\theta, b) | x_b(\theta, b)]$ è $=k+1$. (Questo risultato, nel caso $k=0$, si trova in E. A. Coddington-N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill, New York, 1955 [MR **16**, 1022]; p. 323.)

(β) (Th. V.6) Nelle ipotesi del teorema precedente la varietà $\mathfrak{M}$ definita da (3) è asintoticamente stabile rispetto al sistema (2), con ampiezza asintotica [con ampiezza e fase asintotica] se esistono $h(\rho)$, $g(t)$ continue, $g(t) \to 0$ per $t \to \infty$, ed un $T \geq 0$ tali che $\|X^*(t, x)\| \leq g(t)h(\|x\|)$, $x \in R^n$, $t \geq T$ ed è $\int_T^\infty g(u)\,du < \infty [\int_T^\infty \int_t^\infty g(u)\,du\,dt < \infty]$.

Questi risultati e gli altri, ugualmente interessanti, ma troppo complessi per poterli esporre in dettaglio, contenuti nel lavoro, sono ottenuti attraverso l'introduzione di appropriate coordinate locali in un intorno della varietà in esame e la successiva applicazione ai sistemi ottenuti per la trasformazione di coordinate del teorema del punto fisso di Tychonov [Math. Ann. **111** (1935), 767–776]. Lo studio dei sistemi ottenuti per detta trasformazione (sez. III) è di per sè interessante e si riconnette al problema della stabilità in "spazi prodotto" [S. Lefschetz, *Differential equations: Geometric theory*, Interscience, New York, 1957; MR **20** #1005; p. 117]. *R. Conti* (Florence)

9671:

LaSalle, J. P. Some extensions of Liapunov's second method. IRE Trans. **CT-7** (1960), 520–527.

The present paper is concerned with the application of Lyapunov functions to determine the extent of asymptotic stability and the complete stability of a solution of a system of ordinary differential equations. Except for a theorem of boundedness of solutions, the theoretical results have been previously announced by the author [Proc. Nat. Acad. Sci. U.S.A. **46** (1960), 363–365; MR **22** #3855]. The main point of the present paper is to show that much more information is contained in a Lyapunov function than merely stability and this is clearly illustrated by many examples. {In equation (6), one should have $a > 0$, $b > 0$.} *J. K. Hale* (Baltimore, Md.)

9672:

Staržinskiĭ, V. M. On the Lyapunov method for estimation of a characteristic constant. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk. Meh. Mašinostr. **1959**, no. 4, 46–55. (Russian)

The author considers the real linear differential system

$$\begin{aligned} \frac{dx_1}{dt} &= a_{11}(t)x_1 + a_{12}(t)x_2 \\ \frac{dx_2}{dt} &= a_{21}(t)x_1 + a_{22}(t)x_2 \end{aligned} \tag{1.1}$$

where $a_{ij}(t)$ are piecewise continuous and have period $\omega > 0$. It is also assumed that $a_{12}(t)$ and $a_{21}(t)$ have fixed signs. Write $\int_0^\omega a_{11}(t)dt=\alpha$, $\int_0^\omega a_{22}(t)dt=\beta$. If $\alpha+\beta > 0$, the system is unstable. The author considers the case where $\alpha+\beta \leq 0$ and makes the convention $a=(\alpha-\beta)/\omega \geq 0$.

Introduce the new variables $v=\begin{pmatrix} x \\ y \end{pmatrix}$ where

$$x_1 = x \exp \int_0^t a_{11}(t_1)\,dt_1,\ x_2 = y \exp \int_0^t [a_{22}(t_1)+a]\,dt_1$$

and obtain the transformed differential system

$$\frac{dv}{dt} = B(t)v, \quad B(t) = \begin{pmatrix} 0 & r(t) \\ q(t) & -a \end{pmatrix}, \tag{1.4}$$

where $r(t)$ and $q(t)$ are easily computed explicitly.

Let $V(t)$, with $V(0)=I_2$, be the fundamental matrix solution of (1.4) and consider the Lyapounov constant A, $A=\frac{1}{\omega} \operatorname{sp} V(\omega)=A_0-A_1+A_2-A_3+\cdots$. Here

$$A_0 = \tfrac{1}{2}(1+\exp(-a\omega)),$$

$$A_1 = -\tfrac{1}{2}\int_0^\omega \exp(at_1)[\psi(\omega)-(1-\exp(-a\omega))\psi(t_1)]q(t_1)\,dt_1,$$

where $\psi(t)=\int_0^t \exp(-at_1)r(t_1)dt_1$, and other explicit formulas for A_2, A_3, $\cdots$ are available. The author proves $|A_n/A_{n-1}| < 2|A_1|/n$ and $|A_n/A_{n-1}| < \exp(a\omega)|A_1|/n^2$, for $n \geq 2$. Using these estimates for the ratio $|A_n/A_{n-1}|$, the

author estimates A, in certain cases, and demonstrates the stability of the system (1.1), and the stability of periodic solutions for which (1.1) may be the variational equations.

For instance, if a_{12} and a_{21} have opposite signs and if $A_1 \leqq \frac{1}{2}(1-\exp(-\alpha))(1-\exp\beta)$ then (1.1) is unstable.

If a_{12} and a_{21} have opposite signs and if

$$A_2+\tfrac{1}{2}(1-e^{-\alpha})(1-e^{\beta}) \leqq A_1 \leqq \tfrac{1}{2}(1-e^{-\alpha})(1+e^{\beta})$$

and $0 \leqq \alpha < -\beta$, then (1.1) is stable.

L. Markus (Minneapolis, Minn.)

9673:

Cesari, Lamberto. ★Asymptotic behavior and stability problems in ordinary differential equations. Ergebnisse der Mathematik und ihrer Grenzgebiete. N. F., Heft 16. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1959. vii+271 pp. DM 68.00.

According to the author "the purpose of the present volume is to present many viewpoints and questions in a readable short report . . ."; this objective has been accomplished admirably. In fact, this comprehensive work together with the recent books of R. Bellman [*Stability theory of differential equations*, McGraw-Hill, New York, 1953; MR **15**, 794], E. A. Coddington and N. Levinson [*Theory of ordinary differential equations*, McGraw-Hill, New York, 1955; MR **16**, 1022], hereafter referred to as (C. and L.), and S. Lefschetz [*Differential equations: Geometric theory*, Interscience, New York, 1957; MR **20** #1005], hereafter referred to as (S.L.), provides a rather complete picture of the state of the art up to the present time. Some outstanding features of this book in addition to clarity of presentation are a bibliography of 69 pages, perhaps the most complete in existence up to 1958, carefully cross-referenced with various paragraphs of the book, excellent examples illustrating not only various mathematical concepts, but also physical phenomena (e.g., resonance, relaxation oscillations), and very useful bibliographical notes to further papers in each section. Due to a rather excessive price a detailed description of the contents of the ten sections which comprise the work seems in order.

§ 1. Some remarks on the concept of stability. Basic theorems of existence, uniqueness, continuity of solutions with respect to initial data and parameter for systems of the form (1) $x'=f(t,x)$, where t is real, and x, f are real or complex vectors, are reviewed. Criteria, chiefly those of A. Wintner [Amer. J. Math. **67** (1945), 277–284; **68** (1946), 173–178; MR **6**, 225; **7**, 297] for existence of solutions of (1) on $[t_0, +\infty)$ are given. Definitions of stability, asymptotic stability, conditional stability, all in the sense of Lyapunov, of solutions of (1) and of solutions of normal systems of nth order equations are introduced and illustrated. Boundedness of solutions and the connection with stability in case of linear systems are given, followed by the concepts of asymptotic stability in the large, uniform stability, stability of equilibrium and the theorem of Lagrange. Variational and linear variational systems are introduced and the relation between them, as far as stability investigations is concerned, is discussed. Orbital stability for solutions of autonomous systems, $x'=f(x)$, and the notion of asymptotic phase are introduced, followed by a discussion of "stability and a change of coordinates", and very brief mention of mth order stability in the sense of G. D. Birkhoff.

§ 2. Linear systems with constant coefficients. Relevant matrix theory including norms is reviewed briefly and the Jordan canonical form and logarithm of a matrix, are given. The usual theory, including variation of constants formula, for linear systems, behaviour of solutions of linear homogeneous systems with constant coefficients, Routh-Hurwitz and other criteria are presented. The phase plane is introduced and a complete discussion of linear systems with constant coefficients of order 2 is given. The behaviour of solutions on $[0, \infty)$ of $x'=Ax+f$, where A is a constant matrix and $f(t)$ a given vector is investigated. Applications (linear resonance, servomechanisms, Nyquist stability criterion) are considered.

§ 3. Linear systems with variable coefficients. Gronwall's lemma is used to establish Lyapunov's theorem for the system (1) $x'=(A+C(t))x$, where A is a constant matrix, C a continuous matrix. [If all solutions of $x'=Ax \to 0$ as $t \to +\infty$ and if $\|C(t)\| < c$, c sufficiently small, for $t \geqq t_0$, all solutions of the given system $\to 0$.] Criteria for boundedness of solutions on $[0, +\infty)$ of the scalar equation $x''+f(t)x=0$ are developed (starting with Fatou's erroneous statement) and extended to nth order equations (M. Hukuhara) and to (1) with all solutions of $x'=Ax$ bounded and $C(t)$ measurable, $\int_0^{+\infty} \|C(t)\|dt < \infty$. Further conditions for boundedness are given, e.g., generalized characteristic roots [N. L. Gavrilov, Dokl. Akad. Nauk SSSR **84** (1952), 425–428, 657–660; MR **14**, 275, 276] are introduced and used to prove Cesari's boundedness theorem for the system $x'=(A+B(t)+C(t))x$ [Ann. Scuola Norm. Sup. Pisa (2) **9** (1940), 163–186; MR **3**, 41]. The proof is by reduction to "L-diagonal" form. Theorems, especially those of N. Levinson [Amer. J. Math. **68** (1946), 1–6; Duke Math. J. **15** (1948), 111–126; MR **7**, 381; **9**, 509] and R. Bellman [Duke Math. J. **14** (1947), 83–97; MR **9**, 35], on asymptotic behaviour of solutions of (1) are developed. The concept of linear asymptotic equilibrium is developed. Criteria for uniform and restrictive stability and for reducibility of the system (2) $x'=A(t)x$ are given. Non-homogeneous systems are treated. Lyapunov type numbers are introduced and his theorem (if $A(t)$ is a bounded real continuous matrix on $[t_0, +\infty)$ then every non-zero solution $x(t)$ has a finite type number) is proved. Normal systems of solutions and regular systems of equations are discussed. The paragraph ends with the criterion for kinematic similarity of two matrices $A(t)$ and $B(t)$ continuous and bounded on $[0, +\infty)$ (this is the case if there exists an absolutely continuous nonsingular matrix $L(t)$, bounded together with $L^{-1}(t)$ on $[0, +\infty)$, such that $L^{-1}AL-L^{-1}L'=B$) and the theorem of L. Markus [Math. Z. **62** (1955), 310–319; MR **17**, 37], relating type numbers to generalized characteristic roots.

§ 4. Linear systems with periodic coefficients. Floquet theory for general systems is developed in detail, and applied to the Hill equation, (1) $x''+p(t)x=0$, $p(t+T)=p(t)$, where Lyapunov's boundedness theorem (if $p(t)>0$, $\int_0^T p(t)dt \leqq 4/T$ then all solutions x of (1), together with x', are bounded on $[0, +\infty)$), and several deeper results and extensions are given. Zones of stability and instability are discussed for the scalar equations (2) $x''+\varepsilon p(t)x=0$, (3) $x''+(\delta+\varepsilon r(t))x=0$, where ε, δ are real parameters, and p and r are real nonconstant functions with period T, r of mean-value zero are constructed following O. Haupt [Math. Ann. **79** (1919), 278–285] and the theory is applied to the Mathieu and Meissner equations. The principal

section of this paragraph concerns small periodic perturbations for the system (4) $x' = (A + \varepsilon\Phi(t))x$, where A is a constant $n \times n$ matrix, $\Phi(t)$ has period $2\pi/\omega$, and ε is a small parameter. With the aid of § 3, the easy case where all characteristic values of A have real parts negative, is treated for $|\varepsilon|$ small—this for the almost periodic case of Φ as well. The generalization is given in which $\varepsilon\Phi(t)$ in (4) is replaced by $B(t, \varepsilon)$, where $B(t+T, \varepsilon) = B(t, \varepsilon)$, B is L-integrable in $[0, T]$, continuous in ε for almost all t, $B(t, 0) = 0$, and $\|B(t)\|$ is dominated by an L-integrable function on $[0, T]$. The difficult case where some eigenvalues of A have zero real part is studied in detail by the convergent method of successive approximations developed by L. Cesari [Atti. Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. (6) **11** (1940), 633–695; MR **3**, 41] and modified and extended by L. Cesari, H. R. Bailey, R. A. Gambill, J. K. Hale in more recent papers. (See also § 8 and discussion at end of review.) A number of results, including Cesari's explicit relation for the characteristic exponents of the Floquet theory, are given. A typical theorem is: In (4) let $n = 2$; $\varepsilon\Phi$ is replaced by $B(t, \varepsilon)$ and B satisfies the above assumptions; let $T = 2\pi/\omega$; if the characteristic roots of A are ρ_1, ρ_2, where either (a) $R(\rho_j) < 0$, $j = 1, 2$, or (b) $\rho_1 = i\sigma$, $\rho_2 = -i\sigma$, $\sigma > 0$, $m\omega \neq 2\sigma$, $m = 1, 2, \cdots$, and $\int_0^T \operatorname{tr} B(t, \varepsilon)dt \leqq 0$ for all $|\varepsilon| \leqq \varepsilon_0$; then the absolutely continuous solutions $x(t)$ of (4) are bounded in $[0, +\infty)$ for $|\varepsilon|$ sufficiently small. Extensions to systems of second-order equations and other results are given.

§ 5. The second-order linear differential equation and generalizations. Oscillatory and non-oscillatory solutions for (1) $x'' + f(t)x = 0$ and the more general equation (2) $x'' + g(t)x' + f(t)x = 0$ are studied. Concerning asymptotic behaviour of solutions of (1) when $\lim_{t\to\infty} f(t) = a^2 > 0$, (1) is transformed to (1′) $x'' + (1 - q(t))x = 0$, where $q(t) \to 0$, and the theory developed by G. Fubini [Atti. Accad. Naz. Lincei. Rend. (6) **26** (1937), 253–259] using Liouville's method of successive approximations is presented. The case $f(t) \to -a^2 < 0$ is transferred to (1″) $x'' - (1 + q(t))x = 0$, where $q(t) \to 0$. In case of (1′) it is assumed that $\int^{+\infty} |q(t)|dt < \infty$ and in case of (1″) that $\int^{+\infty} |q(t)| \exp(2t)dt < \infty$. (For problems of this type for systems see (C and L, ch. 3, problems).) When $f(t) \to 0$ as $t \to +\infty$, various sufficient conditions are given which insure that solutions of (1) behave like those of $y'' = 0$ for sufficiently large t. More general theorems of R. Bellman [Ann. Mat. Pura Appl. (4) **31** (1950), 83–91; MR **13**, 132] and G. Prodi [Atti. Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **10** (1951), 447–451; **11** (1951), 30–34; MR **13**, 347, 653] on boundedness and asymptotic behaviour are discussed, followed by a brief résumé for the case $f(t) \to +\infty$ in (1). For the (self-adjoint) equation (3) $Lx + \lambda x = 0$, $Lx = -(px')' + qx$, where λ is a complex parameter, p, q are real-continuous on $[0, +\infty)$, $p > 0$, $p \in c'$, the classical criteria of Weyl for L to be of the limit point or limit circle type at infinity are motivated and stated. Finally, the Parceval relation for functions of class L^2 and some properties of the spectrum of L are discussed. (For a more complete discussion of singular boundary value problems see (C and L, ch. 9).)

§ 6. Some basic theorems on nonlinear systems and the first method of Lyapunov. For the system (1) $x' = Ax + f(t, x)$, where A is a constant n by n matrix with characteristic roots having negative real parts, and f is a vector continuous in (t, x) for $t \geqq 0$ and $x \in \mathcal{U}$, $\mathcal{U}$ a neighbourhood of $x = 0$ in E_n, $\|f(t, x)\| = o(\|x\|)$ as $\|x\| \to 0$ uniformly in t, $0 \leqq t < \infty$, Levinson's proof of the theorem that the solution $x \equiv 0$ is (locally) asymptotically stable is given. Letting $\lambda = \max R(\lambda_i)$, $i = 1, \cdots, n$, where λ_i are the characteristic values of A, it is further shown that the solutions $x(t)$ of (1) for which $\|x(t)\| \to 0$ as $t \to \infty$ have type number $\leqq \lambda < 0$. Various relaxations of the condition $\|f(t, x)\| = o(\|x\|)$ are pointed out, and the instability theorem (at least one λ_i, say λ_j, has $\operatorname{Re} \lambda_j > 0$) is stated [for proofs see (C. and L., ch. 13)]. The extension of the above results to the case where A in (1) is replaced by $A(t)$, where $A(t)$ is a periodic matrix, is given. The theorem on conditional stability for (1) (existence of a stable manifold) is stated. [For proofs as well as precise behaviour solutions of the stable manifold see (C. and L.) ch. 13]. Extensions due to M. Golomb [Arch. Rational Mech. Anal. **1** (1958), 272–282; MR **21** #2777], of the Dini-Hukuhara theorems for linear systems, to the case of (1) with A replaced by $A(t)$ and the system $y' = A(t)y$ restrictively or uniformly stable, are stated and proved. Stability criteria for periodic solutions of nonautonomous periodic systems $x' = F(t, x)$ are given using the first variation. The notions of harmonic, subharmonic, superharmonic, ultra-subharmonic solutions are defined and the results J. L. Massera [Duke Math. J. **17** (1950), 457–475; MR **12**, 705] relating, e.g., the existence of subharmonic and harmonic solutions are given. Periodic solutions of autonomous systems are discussed briefly and the theorem on orbital stability and the existence of an asymptotic phase for a periodic solution is stated. The Lyapunov method of successive approximations (called the First Method) for the system $x' = A(t)x + P(t, x)$, where $A(t)$ is a bounded continuous matrix on $[t_0, +\infty)$ and the components of P are analytic in $x_1, \cdots, x_n$, $t \in [t_0, +\infty)$, and the power series begin with quadratic terms, is given in detail for both the finite and infinite interval. Stability "beyond the greatest type number" is discussed; cf. R. F. Bylov [Dokl. Akad. Nauk SSSR **103** (1955), 181–184; MR **17**, 37]. Theorems of R. Bellman [Ann. of Math. (2) **49** (1948), 515–522; MR **10**, 121] concerning existence of bounded, L-integrable, or L^2-integrable solutions on $[0, +\infty)$ for the system $x' = A(t)x + \Phi(t, x)$ are treated. The theorem on measure-preserving transformations defined by solutions of the autonomous system $x' = F(x)$ is proved, and its importance for Hamiltonian systems is discussed. The notion of an invariant set and stability in the sense of Poisson are defined, followed by a brief discussion of recurrency in the sense of G. D. Birkhoff. Finally, differential equations on the torus are studied briefly.

§ 7. The second method of Lyapunov. Detailed proofs of the basic stability theorems using this method, and the theorem on instability, for the trivial solution of the system (1) $x' = f(t, x)$, are given. Equi-asymptotic stability is defined and some recent results, with a brief reference to converse theorems, are discussed. See also (S. L.); for a very recent complete survey of the Second Method, see H. A. Antosiewicz [*Contributions to the theory of nonlinear oscillations, Vol. IV*, pp. 141–166, Princeton Univ. Press, Princeton, N.J., 1958; MR **21** #1432] and for more recent results T. Yoshizawa [Mem. Coll. Sci. Univ. Koyto. Ser. A. Math. **32** (1959), 171–180; and Funkcial. Ekvac. **2** (1959), 95–142; MR **21** #5789; **22** #5789]. Finally, by considering a particular first-order partial differential equations, new proofs of the simplest stability and instability

theorems considered in § 6 are given; however, here the autonomous case is taken; $x' = Ax + P(x)$, where P is analytic in $x_1, \cdots, x_n$, and the power series begin with quadratic terms.

§ 8. Analytic methods. Here the existence and approximation of periodic solutions of nonlinear (especially weakly nonlinear) systems are studied by various methods beginning with that of Lindsted where "small divisors" may occur. The method of Poincaré is discussed in detail for the scalar equation $x'' + x = \varepsilon f(x, x')$, where ε is a small parameter, and the appearance of secular terms is clearly brought out. The extension of this method to general real systems of the form $x' = Ax + \mu f(t, x, \mu)$, and also the autonomous case, where A has at least one characteristic root of the form iN, N an integer or zero, and f has period 2π in t, due to Coddington and Levinson (C. and L., ch. 14), is referred to only in the bibliography. A recent paper by the reviewer gives criteria for stability and instability of the periodic solutions of such systems [J. Reine Angew. Math. **203** (1960), 64–79; MR **22** #8725]; also Y. Sibuya [M.R.C. Tech. Sum. Rep. No. 102, Univ. of Wis., August, 1959]. The method of Krylov, Bogolyubov, and Van der Pol is presented for the equation $x'' + \sigma^2 x + \varepsilon f(x, x') = 0$. It is stated that the convergence of the method of successive approximations has not been established. {The reviewer understands that the proof is implicit in recent work of Bogolyubov and Mitropolsky [*Asimptotičeskie metody v teorii nelineĭnyh kolebaniĭ*, 2nd ed., Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1958; MR **20** #6812].} The convergent method developed by L. Cesari and modified in a series of papers by L. Cesari, J. K. Hale, H. R. Bailey, R. A. Gambill, W. R. Fuller, E. W. Thompson, including some criteria for existence of subharmonic oscillations and for stability, is presented in detail for the analytic case only. [See further remarks below.] The classical perturbation method of Poincaré (C. and L., ch. 14, and S. L.) for both nonautonomous and autonomous systems, including criteria for stability, instability and orbital stability, is discussed; this method cannot be applied to the above problems due to the vanishing of certain Jacobians. The theorem of N. Levinson and O. K. Smith [Duke Math. J. **9** (1942), 382–403; MR **4**, 42] concerning the existence and uniqueness of a periodic solution of the Liénard equation $x'' + f(x)x' + g(x) = 0$ is proved with great care, followed by an oscillation theorem and an existence theorem of D. Graffi [Mem. Accad. Sci. Bologna (9) **7** (1940), 121–129; MR **9**, 589] for the more general equation $x'' + f(x, x')x' + g(x) = 0$. Nonlinear free oscillations for $x'' + f(x) = 0$, $f(0) = 0$, $f(x)$ odd, are studied in detail. The theorem of N. Levinson [Ann. of Math. (2) **52** (1950), 727–738; MR **12**, 335] is discussed, on the existence of an invariant surface, $|\varepsilon|$ sufficiently small, for the perturbed system $x' = F(x) + \varepsilon g(t, x, x')$, F, G real and sufficiently smooth, where $x' = F(x)$ has a strongly stable solution of period 2π, and where g has period T in t; see also very recent work of W. Loud [Ann. of Math. (2) **70** (1959), 490–529; MR **21** #5786] and J. K. Hale [ibid. **73** (1961), 496–531]. The paragraph closes with an excellent discussion of nonlinear resonance, various prime movers, relaxation oscillations.

§ 9. Analytical-topological methods. The Poincaré theory of isolated critical points is presented in particular for the real autonomous system ($n = 2$) (1) $x' = Ax + f(x)$, where A is a constant matrix $\det A \neq 0$, $\|f\| \leq \psi(r)$, $r = (x_1^2 + x_2^2)^{1/2}$, $\psi(r) = o(r)$ as $r \to 0$, based on the development of A. Wintner [Amer. J. Math. **69** (1947), 815–824; MR **9**, 285] and (C. and L., ch. 15). This is followed by an excellent exposition of the Poincaré-Bendixson theory and extensions, in particular to nth order systems, and the concepts of total stability and structural stability. N. Levinson's theorem [J. Math. Phys. **22** (1943), 41–48; MR **5**, 66] is proved with great care; it concerns the Liénard equation (2) $x'' + f(x, x')x' + g(x) = e(t)$, and asserts the existence of a closed curve C in the phase plane enclosing the origin and some conveniently large circle, with the property that every solution $x = x(t)$, $x' = x'(t)$ crossing C passes from the exterior of C to the interior. The existence of at least one nonconstant periodic solution for (2) with $e(t) \equiv 0$, due to Levinson (op. cit.), is proved, and this theorem is combined with the one above to obtain a number of other results, due essentially to Levinson and Smith, for (2) with $e(t) \equiv 0$. The Brouwer fixed-point theorem is then used to obtain the existence of a periodic solution for (2) with $e(t)$ of period T. Here the closed curve C constructed above plays an essential role. (For very recent results on periodic solutions of $x'' + cx' + g(x) = e(t)$, see also W. S. Loud [Mem. Amer. Math. Soc. No. 31 (1959); MR **21** #5785]; for other applications of fixed point theorems see remarks below and [#9664].) The method of M. L. Cartwright, requiring refinements of the Brouwer theorem, is presented. The paragraph closes with an account of the essentially topological method of Ważewski [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **3** (1947), 210–215; MR **9**, 512].

§ 10. Asymptotic developments. This is a brief account of Poincaré's concept of asymptotic development, the theorem on asymptotic expansions of solutions of linear systems having an irregular singular point of finite order at infinity, and asymptotic developments deduced from Taylor expansions. Systems with a large parameter are considered, turning points are defined and principal results for the case of no turning points are stated. For recent results see also Y. Sibuya [J. Fac. Sci. Univ. Tokyo. Sect. I **7** (1958), 527–540; MR **20** #2514]. The method of R. E. Langer giving a constructive procedure for complete asymptotic expansions valid in a full neighbourhood of a simple turning point is described for second-order equations, and some extensions chiefly, those of R. E. Langer and his students, are pointed out. For very recent results see also R. E. Langer [Trans. Amer. Math. Soc. **90** (1959), 113–142; MR **21** #4270], C. C. Lin and A. L. Rabenstein [M.R.C. Tech. Sum. Rep. No. 37, August, 1958, Univ. of Wis.]; W. Wasow [J. Math. and Phys. **38** (1959/60), 257–278; MR **22** #5787]; A. L. Rabenstein [Arch. Rational Mech. Anal. **1** (1958), 418–435; **2** (1959), 355–366; MR **20** #3337; **21** #2448]; Y. Sibuya [M.R.C. Tech. Sum. Rep. No. 105, Dec. 1959, Univ. of Wis.]; H. L. Turritin and W. A. Harris, Jr. [Tech. Rep. No. 2, Univ. of Minn., Dec. 13, 1957, DA-11-022-ORD-2042]. The final section gives a brief account of singular perturbation problems following essentially W. Wasow [*Contributions to the theory of nonlinear oscillations*, pp. 313–350, Princeton Univ. Press, Princeton, N.J., 1950; MR **12**, 29]. For recent results see also J. J. Levin [Trans. Amer. Math. Soc. **85** (1957), 357–368; Duke Math. J. **24** (1956), 609–620; MR **19**, 548; **18**, 305]; N. Levinson [Duke Math. J. **25** (1958), 331–342; MR **21** #165]; S. Haber and N. Levinson [Proc. Amer. Math. Soc. **6** (1955), 866–872; MR **17**, 618]; L. Flatto and N. Levinson [J. Rational Mech. Anal. **4** (1955), 943–950; MR **17**, 849],

and Wm. A. Harris, Jr. [Tech. Rep. No. 3, Univ. of Minn., March 20, 1958, DA-11-022-ORD-2042].

Little was said above about the important method of L. Cesari. The recent, very clear formulation of the method, which is due partly to L. Cesari [as presented at the Internat. Conf. Differential Equations, Mexico City, Sept., 1959] and partly to W. R. Fuller [Thesis, Purdue Univ., PRF 1398, 1957], is partly implicit in the papers of L. Cesari, J. K. Hale et al. and deserves at least a brief summary, for in the opinion of the reviewer it constitutes an important tool which might be generalized to handle even more complicated problems; e.g., the case where the unperturbed system is not linear. Consider the system (1) $x' = Ax + \varepsilon q(t, x, \varepsilon)$ (alternatively the autonomous case) where q has period $T = 2\pi/\omega$ in t and is L-integrable on $[0, T]$, ε is a small parameter, A is a constant $n \times n$ matrix; for simplicity let $A = \operatorname{diag}[\rho_1, \cdots, \rho_n]$ where ρ_j may depend on ε continuously and $\rho_j(0) = i\tau_j = ia_j\omega/b_j$, a_j, b_j integers. For $\varepsilon = 0$, (1) has a solution of the form $z(t) = [c_1 \exp(i\tau_1 t), \cdots, c_n \exp(i\tau_n t)]$, where c_j are constant. Define $B = A(0)$ and let Ω be the space with uniform topology of continuous vectors $\phi = (\phi_1, \cdots, \phi_n)$ of period $2\pi b/\omega$, $b = \prod_{i=1}^{n} b_i$, with $m[\phi_j(t) \exp(-i\tau_j t)] = c_j$; here m is the mean value. Define the transformation $\psi = T\phi$ of Ω into itself by (2) $\psi(t) = z(t) + \varepsilon \exp(Bt) \int \exp(-Bu) \times \{q(u, \phi(u), \varepsilon) - \phi(u) D[\phi]\} du$, where $D = \operatorname{diag}\{d_1, \cdots, d_n\}$ defined by $c_j d_j = m\{\exp(-i\tau_j t) q_j[t, \phi(t), \varepsilon]\}$, $j = 1, \cdots, n$. Note that the integrand is periodic and has mean value zero, and $\psi \in \Omega$ if ϕ does, i.e., $T\Omega \subset \Omega$. Under suitable and very mild conditions on q, for $|\varepsilon|$ sufficiently small, the Schauder fixed-point theorem may be used to deduce at least one solution of the integral equation

$$x(t) = z(t) + \varepsilon \exp(Bt) \int \exp(-Bu)\{q[u, x(u), \varepsilon] - x(u) D[x]\} du,$$

and thus also of the differential equation (3) $x' = [B - \varepsilon D]x + \varepsilon q(t, x, \varepsilon)$. If now (4) $B - \varepsilon D = A$, then $x(t)$ is a periodic solution of (1). Analysis of (4) and of T thus leads to existence theorems for periodic solutions or families of periodic solutions for (1). Moreover, if q is Lipschitzian in the components of x, T is a contraction whose fixed point may be uniformly approached by the successive approximations of the form $x^{(0)}(t) = z(t)$, $x^{(m)}(t) = Tx^{(m-1)}(t)$, $m = 1, 2, \cdots$. {The reviewer has tried to connect the above method with that of Coddington and Levinson (C. and L., ch. 14) and he feels that while similar problems may be treated by each, they appear to be quite different.}

Finally it would be unfair not to point out that this fine book is not without a number of misprints which the reader should have little difficulty ironing out. While the reviewer regrets that it was necessary to treat some topics (e.g., the second method, Hamiltonian systems, and singular perturbation theory) rather briefly, he feels that this excellent book should be in the hands of everyone doing research in this and related fields.

J. A. Nohel (Atlanta, Ga.)

9674:

Chin, Yuan-Shun. Theory of regional analysis of ordinary differential equations. II. Acta Math. Sinica **6** (1956), 184–205. (Chinese. English summary)

Continuing his previous work of regional analysis [same Acta, 19–34; MR 18, 127], the author further classifies in more detail the points in the phase plane according to the linear approximation (in the mean) of the directional vector of the trajectories. He then applies this to the problems involving small parameters, such as Lighthill's example [Philos. Mag. (7) **40** (1949), 1179–1201; MR **11**, 518] by constructing the trajectories in the large in order to explain the difficulty of the divergence of the expansion of the solutions in terms of the power series of the small parameter. Finally, he uses his method to investigate the location of limiting cycles and applies it to several famous equations. *S. S. Shu* (Lafayette, Ind.)

9675:

Chin, Yuan-Shun. Theory of regional analysis of ordinary differential equations. III. Acta Math. Sinica **6** (1956), 363–373. (Chinese. English summary)

The author deals with the equation $\ddot{x} + P(\dot{x}) + Q(x) = 0$, where $P(y)$ is a polynomial with a simple zero at $y = 0$ (all other zeroes are pure imaginary), and $Q(x)$ is either a polynomial with simple zeroes or some other similar function. The straight lines determined by $Q'(x) = 0$ divide the phase plane into regions, containing one singular point each. From the behavior of the singular point, he determines the topological structure of the trajectories in each of such regions and thus in the whole plane. *S. S. Shu* (Lafayette, Ind.)

9676:

Corduneanu, C. On the existence of bounded solutions for some non-linear differential systems. Dokl. Akad. Nauk SSSR **131** (1960), 735–737 (Russian); translated as Soviet Math. Dokl. **1**, 319–321.

Perron [Math. Z. **32** (1930), 465–473] has proved certain theorems on stability for perturbed linear triangular differential systems provided that the diagonal elements satisfy given properties. The present paper generalizes these ideas in various directions for the case where the unperturbed equation is nonlinear. The proofs are not given but it is fairly clear how they would be obtained. In the fourth line of the paper, the word "bounded" should be replaced by "unbounded", a mistake which was made in the English translation.

J. K. Hale (Baltimore, Md.)

9677:

Trevisan, Giorgio. Un teorema di stabilità per le equazioni differenziali lineari del secondo ordine. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **28** (1960), 467–469.

The author proves: If $\alpha(x)$, $\beta(x)$, and $Q(x)$ are continuous on $[a, \infty)$, and if $\alpha(x)$ and $\beta(x)$ are positive, and if $\alpha(x)$ and $\beta(x) \cdot \exp 2\int_{x_0}^{x} Q(t)dt$ are non-decreasing, then, for any solution $y(x)$ of $y'' + Qy' + \beta y/\alpha = 0$, $(y'(x))^2 \leq \beta(x) \cdot C$ and $y^2(x) \leq \alpha(x) \cdot C$ (C is constant). Hence, y and y' are bounded on $[a, \infty)$ if α and β are bounded.

W. J. Coles (Salt Lake City, Utah)

9678:

Bogdanov, Yu. S. On the existence of an approximative sequence for a regular linear differential system. Uspehi Mat. Nauk **15** (1960), no. 1 (91), 177–179. (Russian)

Consider a system (1) $dx/d\tau = px$, where p is a (2×2) real matrix, piecewise continuous and bounded for $\tau \geq 0$. Let T be an arbitrary increasing sequence of real numbers

$t_k \to \infty$ as $k \to \infty$, and let p_k be a matrix $= p$ for $0 \leqq \tau < t_k$ and continued periodically beyond t_k. Denote by $\lambda = (\lambda^1, \lambda^2)$, $\lambda_k = (\lambda_k{}^1, \lambda_k{}^2)$ the characteristic exponents of the systems (1) and $dx/d\tau = p_k x$, respectively. The sequence T is called approximative if $\lambda_k \to \lambda$ as $k \to \infty$. R. E. Vinograd [Uspehi Mat. Nauk **9** (1954), no. 2 (60), 125–128; MR **16**, 360] showed that for some regular [in the sense of Lyapunov, *Problème général de la stabilité du mouvement*, Princeton Univ. Press, Princeton, N.J., 1947; MR **9**, 34] systems (1), an arbitrary sequence T need not be approximative. The author shows that there exists a regular system (1) for which no sequence T is approximative and proves that by an orthogonal transformation of variables x_1, x_2 depending only on τ, any regular system (1) can be transformed into a system for which an approximative sequence exists. *Z. Opial* (Kraków)

9679:

Chang, Li-ling. Topological structure of integral curves of the differential equation

$$\frac{dy}{dx} = \frac{ax^2 + bxy + cy^2 + \varphi(x, y)}{ex + fy + \psi(x, y)}$$

(when $\varphi = \psi = 0$ the fraction is irreducible) in the neighborhood of the origin. Advancement in Math. **3** (1957), 650–654. (Chinese)

For the case $e > 0$, the author finds the criteria for the topological structure of integral curves in the neighborhood of the origin. *S. S. Shu* (Lafayette, Ind.)

9680:

Chang, Die. Topological structure of integral curves of the differential equation

$$\frac{dy}{dx} = \frac{ax^3 + bx^2y + cxy^2 + dy^3}{a_1x^3 + b_1x^2y + c_1xy^2 + d_1y^3}.$$

Advancement in Math. **3** (1957), 234–245. (Chinese. English summary)

Author's summary: "The topological structure of integral curves are classified into fifteen classes with algebraic criteria." *S. S. Shu* (Lafayette, Ind.)

9681:

Proskuriakov, A. P. Construction of periodic solutions of autonomous systems with one degree of freedom in the case of arbitrary real roots of the equation for the basic amplitudes. J. Appl. Math. Mech. **22** (1958), 709–720 (510–518 Prikl. Mat. Meh.).

The method of construction of periodic solutions for the autonomous scalar equation (1) $\ddot{x} + k^2x = \mu f(x, \dot{x}, \mu)$ where f is analytic with respect to all its arguments, k is a real constant and μ is a small parameter (see I. G. Malkin [*Nekotorye zadači teorii nelinejnyh kolebanií*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956; MR **18**, 396; also available as AEC Transl. 3766, Books 1, 2] and the author [Prikl. Mat. Meh. **21** (1957), 585–590; MR **19**, 1083]) breaks down if the equation for the determination of basic amplitudes has multiple roots. The author gives a systematic procedure for the determination of periodic solutions of (1) in this case, thus extending the method of Poincaré; see, e.g., the recent book of L. Cesari [#9673]. Roughly speaking, the method involves obtaining of further conditions from the Poincaré periodicity conditions by using more terms of the power series expansions. Several useful examples are given.

J. A. Nohel (Atlanta, Ga.)

9682:

Le Corbeiller, P. Two-stroke oscillators. IRE Trans. **CT-7** (1960), 387–398.

Given a system of n first-order real autonomous ordinary differential equations $dx/dt = f(x)$, where $x = (x_1, \cdots, x_n)$ and $f = (f_1(x), \cdots, f_n(x))$ and a real-valued function $E = E(x)$ (e.g., energy if the x variables are phase-space variables), the author defines the 'strokes' of an oscillation (periodic solution) as the number of maxima and minima of $E(x)$ on that oscillation. The author exhibits several examples and then establishes (by arguments which are less casual than he claims) that they have two-stroke oscillations. *S. P. Diliberto* (Berkeley, Calif.)

9683:

Mufti, I. H. Stability in the large of systems of two equations. Arch. Rational Mech. Anal. **7** (1961), 119–134.

The asymptotic stability in the large of the solution $x = y = 0$ is proved for 2-dimensional autonomous systems of the following three forms: (1) $x' = h_1(y)x + ay$, $y' = h_2(x)x + by$, (2) $x' = h_1(y)x + ay$, $y' = bx + h_2(x)y$, (3) $x' = F(x, y)$, $y' = f(\sigma)$. In (3), $\sigma = c_1x - d_1y$, where c_1, d_1 are positive constants. The main assumption appearing in all results for (1) is (4) $b + h_1(y) < 0$, $bh_1(y) - ah_2(x) > 0$ for all $x \neq 0$, $y \neq 0$, and for (2) is (5) $h_1(y) + h_2(x) < 0$, $h_1(y)h_2(x) - ab > 0$. However, asymptotic stability in the large is proved under some restrictions in addition to (4) (or (5)). For example, in (1), four cases are considered: (i) $ah_2(x) < 0$, $h_1(y) < 0$, $b < 0$; (ii) $ah_2(x) > 0$; (iii) $b > 0$ or $h_1(y) > 0$; and (iv) $b = 0$.

Assumptions under which the asymptotic stability in the large holds for (3) are: $F(0, 0) = 0$, $\partial F/\partial y < 0$, $f(0) = 0$, $f(\sigma) \neq 0$ for $\sigma \neq 0$, $f'(\sigma) \geqq 0$ and either one of the following conditions $\partial F/\partial x < 0$ or $\partial F/\partial x = 0$. Also discussed is the case $\partial F/\partial x > 0$. In this case additional requirements are made to the effect that $x = y = 0$ is asymptotically stable in the small and (3) admits neither a periodic solution nor a singular point other than $x = y = 0$. In the case (iii) of (1) the author also requires that $x = y = 0$ is locally asymptotically stable. This seems to be unnecessary and, in the reviewer's opinion, may be concluded from the other assumptions. In the proofs of Theorems 4.1 and 4.2 dealing with (3), the author makes use of the fact that $f'(0) > 0$, which is not explicitly assumed.

C. Olech (Baltimore, Md.)

9684:

Vorel, Zdeněk. Some estimates in the theory of a quasi-linear system with one degree of freedom. Apl. Mat. **6** (1961), 1–24. (Czech. Russian and English summaries)

In the first paragraph the author examines the equation (1) $\ddot{x} + x = \varepsilon f(x, \dot{x}, t, \varepsilon)$ where ε is a small positive parameter and $f(x_1, x_2, t, \varepsilon)$ is supposed to be continuous and to satisfy a Lipschitz condition with respect to x_1 and x_2. He transforms (1) by means of polar coordinates a, ψ into the system (2) $\dot{a} = -\varepsilon f(a \cos\psi, -a \sin\psi, t, \varepsilon) \sin\psi$, $\dot{\psi} = 1 - \varepsilon a^{-1} f(a \cos\psi, -a \sin\psi, t, \varepsilon) \cos\psi$ and, using the

method of successive approximations, he gets approximations $a_m(t)$ and $\psi_m(t)$ for $a(t)$ and $\psi(t)$ and proves the estimates $a(t)-a_m(t)=O(\varepsilon^{m+1})$, $\psi(t)-\psi_m(t)=O(\varepsilon^{m+1})$ for $0\leq t\leq T$. Moreover the estimate of the period of one revolution of the solution of (1) around the origin in the $(x, \dot{x})$-plane is given. In the remaining part of the paper the author studies the autonomous case (3) $\ddot{x}+x=\varepsilon f(x, \dot{x}, \varepsilon)$ and proves by means of the preceding results the existence of a positively stable periodic orbit of (3) and estimates its position and period. *M. Zlámal* (Brno)

9685:

Miščenko, E. F.; Pontryagin, L. S. Derivation of certain asymptotic estimates for solutions of differential equations with a small parameter in the derivatives. Izv. Akad. Nauk SSSR. Ser. Mat. **23** (1959), 643–660. (Russian)

Pontryagin derived formal asymptotic approximations for solutions of the system $\varepsilon\dot{x}^i=f^i(x^1, \cdots, x^k, y^1, \cdots, y^l)$, $\dot{y}^j=g^j(x^1, \cdots, x^k, y^1, \cdots, y^l)$ $(i=1, \cdots, k;\ j=1, \cdots, l)$ in the neighborhood of an exceptional point, that is, where $\det\|\partial f^i/\partial x^j\|=0$ [same Izv. **21** (1957), 605–626; MR **20** #1029a]. The proof that these approximations are exact to the expected degree of accuracy is now given. The system is transformed to, and treated in, the form (*) $d\xi^i/d\xi^1=\Phi^i(\xi, \eta)/\Phi^1(\xi, \eta)$, $d\eta^j/d\xi^1=\varepsilon\Psi^j(\xi, \eta)/\Phi^1(\xi, \eta)$ $(i=2, \cdots, k;\ j=1, \cdots, l)$. The interval of ξ^1 is $[-p, p]$ where p is small and independent of ε. Three cases are considered: $-p\leq\xi^1\leq-\varepsilon^{2/7}$, $-\varepsilon^{2/7}\leq\xi^1\leq\varepsilon^{2/3}$, $\varepsilon^{2/3}\leq\xi^1\leq p$. The idea of the proof in each of these cases is to surround the formal approximation to a solution of (*) by a narrow closed neighborhood U, a "pipe", whose radius converges to zero as a positive power of ε as $\varepsilon\to0$. It is shown that if the initial point of a solution is taken in U, then the solution remains in U. To this end, U is constructed so that some of its "walls" are Lyapunov functions associated with the system (*) when it is written with left-hand members $\varepsilon\dot{\xi}^1$, $\varepsilon\dot{\xi}^i$, and $\dot{\eta}^j$ [L. S. Pontryagin, *Lekcii po obyknovennym differencial'nym uravneniyam*, Izdat. Moskov. Gos. Univ., Moscow, 1956].

N. D. Kazarinoff (Moscow)

9686:

Mishchenko, E. F.; Pontryagin, L. S. Differential equations with a small parameter attached to the higher derivatives and some problems in the theory of oscillation. IRE Trans. **CT-7** (1960), 527–535.

Heuristic considerations, along with the statement (with references to proofs) of some theorems, concerning solutions of a system $\varepsilon\dot{x}=f(x, y)$, $\dot{y}=g(x, y)$ in which x, y denote vectors and ε is a small parameter. For example: (1) a trajectory may consist of stretches on which $f(x, y)$ is approximately zero, joined by stretches on which y is approximately constant and x varies rapidly; (2) if the x-system has, for constant y, a stable limit cycle on which x has the mean value $\bar{x}(y)$, a solution may be approximated via the solution of $\dot{y}=g(\bar{x}(y), y)$.

T. M. Cherry (Melbourne)

9687:

Tung, Čin-ču. Distributions of indices of the singular points and positions of the limit cycles of a system of the differential equations

$$\begin{aligned}\frac{dx}{dt}&=X_2(x, y)=\sum_{0\leq i+k\leq2}a_{ik}x^iy^k\\ \frac{dy}{dt}&=Y_2(x, y)=\sum_{0\leq i+k\leq2}b_{ik}x^iy^k\end{aligned}\qquad(E_2).$$

Acta Math. Sinica **8** (1958), 258–268. (Chinese. Russian summary)

Elaboration of the treatment of indices and limiting cycles of the Petrovskiĭ-Landis equation [Dokl. Akad. Nauk SSSR **102** (1955), 29–32; MR **16**, 1110]. The author shows that every closed characteristic contains only one critical point and is always convex, by a method similar to Poincaré-Bendixson's theory. He further gives an example in which there are only two real finite critical points and shows that it has two limiting cycles. Certain geometrical configurations (such as in the case of three critical points all with the index $+1$, the configuration of the limiting cycles containing one critical point each and mutually exterior to one another) of three limiting cycles (the maximum number according to Petrovskiĭ-Landis) are impossible. *S. S. Shu* (Lafayette, Ind.)

9688:

Tung, Čin-ču. The position of limit cycles of the system of differential equations

$$\frac{dx}{dt}=\sum_{0\leq i+k\leq2}a_{ik}x^iy^k,\qquad\frac{dy}{dt}=\sum_{0\leq i+k\leq2}b_{ik}x^iy^k.$$

Acta Math. Sinica **9** (1959), 156–169. (Chinese. Russian summary)

For certain polynomials $P(x, y)$ and $Q(x, y)$ of second degree, the author proves in this paper that the differential system $dx/dt=P$, $dy/dt=Q$ has three limit cycles L^1, L^2, L^3 whose positions satisfy $I_1\cap I_2\neq0$, $I_1\cap I_3=0$, where I_1, I_2 and I_3 are the bounded component of the complement of L^1, L^2 and L^3 respectively. This paper is a continuation of a previous one [#9687]. The author has also written another paper in English [Sci. Sinica **8** (1959), 151–171; MR **21** #5040], an abridged and revised version of the above two articles, which describes completely the possible relative positions of the limit cycles of the above differential system. *Choy-tak Taam* (Washington, D.C.)

9689:

Liu, Yung-ching. On differential equation with algebraic limit-cycle of second degree

$$\frac{dy}{dx}=\frac{a_{10}x+a_{01}y+a_{30}x^3+a_{21}x^2y+a_{12}xy^2+a_{03}y^3}{b_{10}x+b_{01}y+b_{30}x^3+b_{21}x^2y+b_{12}xy^2+b_{03}y^3}.$$

Advancement in Math. **4** (1958), 143–149. (Chinese. English summary)

The author considers the existence, uniqueness and stability of a limiting cycle which is an ellipse.

S. S. Shu (Lafayette, Ind.)

9690:

Lakshmikanth, V. On the asymptotic connections between the solutions of the differential systems. Bul. Inst. Politehn. Iaşi (N.S.) **5 (9)** (1959), no. 3–4, 21–24. (Russian and Romanian summaries)

The systems of differential equations considered are those of the form

$$x' = ax+by+f_1(x, y, t),$$
$$y' = cx+dy+f_2(x, y, t),$$

where a, b, c, d are real, $ad-bc\neq 0$, and f_1 and f_2 are real functions defined in the region $0\leq t<\infty$, $r<\infty$ of the polar plane. Using a lemma of an earlier paper [Proc. Amer. Math. Soc. 8 (1957), 1044–1048; MR **20** #3342] the author proves the following generalization of a theorem of Wintner [Amer. J. Math. **69** (1947), 815–824; MR **9**, 285]. Suppose that (0, 0) is a vortex of the linear part $x'=ax+by$, $y'=cx+dy$, of the system; that $h(r, t)\geq 0$ is measurable in t for fixed $r\geq 0$, continuous and non-decreasing in r for fixed t on $t_0\leq t<\infty$, $r\geq 0$; that $r(t)$ is the maximum solution of $r'=\varepsilon^t h(r, t)$ through $(t_0, 0)$ satisfying $r(t)=o(1)$ as $t\to\infty$; and that the functions f_1, f_2 satisfy $|f_i(x, y, t)|\leq h(r, t)$, $i=1, 2$. Then every solution of the non-linear system tending to the vortex (0, 0) is asymptotic to a solution to the linear system and vice versa. *W. R. Utz* (Columbia, Mo.)

9691:

Mel'nikov, V. K. Determination of the region of capture for almost conservative second-order equations. Mat. Sb. (N.S.) **49 (91)** (1959), 353–380. (Russian)

This paper contains a study of nonlinear differential equations of the form

$$(*) \qquad \frac{d}{dt}(m(\varepsilon, t)\dot{x})+k(\varepsilon, t)p'(x) = \varepsilon f(\varepsilon, t, x, x)\dot{x}$$

which for $\varepsilon=0$ describe nonlinear conservative motions since $k(0, t)$, $m(0, t)$ are assumed to be independent of t. The right-hand side supplies a small amount of friction. The problem discussed consists in the determination of the regions of capture defined as follows. Let x_r be an equilibrium solution of (*). Then the region of capture with respect to x_r at $t=t_0$ consists of those points x_0, $\dot{x}_0$ in the phase plane for which the solutions of (*) with the initial conditions $t=t_0$, $x=x_0$, $\dot{x}=\dot{x}_0$ perform stable oscillations about x_r. In the conservative case different regions of capture are separated by solutions (called separatrices) which are asymptotic to equilibrium solutions (take for orientation the example $\ddot{x}+\sin x=0$). The regions of capture are determined by the separatrices which are described by asymptotic expansions in terms of powers of ε.

This problem arises in the study of phase oscillations in particle accelerators where region of capture corresponds to the particles which are accelerated.

J. Moser (New Rochelle, N.Y.)

9692:

Kazarinoff, N. D.; Ritt, R. K. Scalar diffraction theory and turning-point problems. Arch. Rational. Mech. Anal. **5**, 177–186 (1960).

This paper considers the initial value problem $\nabla^2 u-u_{tt}=\rho(x)e^{i\omega t}$, $u(x, 0)=f(x)$, $u_t(x, 0)=g(x)$, $u_n(x, t)=0$, where $\rho(x)$, $f(x)$ and $g(x)$ are functions of compact support in the region exterior to a smooth, closed bounded surface in $E^{(3)}$, and u_n is the normal derivative at the surface. The main theorem asserts that the function $v=ue^{-i\omega t}$ has a $(C-1)$ limit, $\tilde{v}$, which satisfies the scalar wave equation $(\nabla^2+\omega^2)\tilde{v}=0$ and that $\tilde{v}$ is $\lim_{s\to 0}\phi(x, s)$, where ϕ satisfies the equation $[\nabla^2+(\omega+is)^2]\phi=0$, $s>0$; ϕ lies in the L^2 space of the exterior region, and satisfies the boundary condition. {The reviewer has been informed by the authors that the proof of lemma 3 is incorrect, and that a correct proof exists.} This theorem permits the application of the resolvent theory of Sims [J. Math. Mech. **6** (1957), 247–285; MR **19**, 144] and Phillips [Mathematics Research Group, Washington Square College of Arts and Science, N.Y.U., Res. Rep. No. EM-42 (1952); MR **14**, 1088] to secure the alternative representation for the Green's function (analagous to the well known Watson transform) in axially symmetric diffraction problems. The analytic problem centers about the turning point analysis of ordinary second-order differential equations. In the case of an ellipsoid of revolution of eccentricity close to unity the question reduces to the study of two turning points, close together. This is, for the present, an unsolved problem. *A. E. Heins* (Ann Arbor, Mich.)

9693:

Volosov, V. M. The method of averaging. Dokl. Akad. Nauk SSSR **137** (1961), 21–24 (Russian); translated as Soviet Math. Dokl. **2**, 221–224.

The author considers the system (1) $x'=\varepsilon X(x, y, t, \varepsilon)$, $y'=Y(x, y, t, \varepsilon)$ ($'=d/dt$), where x and y are vectors of dimension n and m respectively. The solution of the degenerate system (2) $y'=Y(x, y, t, 0)$, $x=\text{const}$ is assumed known as $y=\varphi(x, y_0, t_0, t)$, where $\varphi(x, y_0, t_0, t_0)=y_0$. It is assumed that the right members in (1) have mean values with respect to t. The x-equation is averaged to give

$$(3) \quad \bar{x}' = \varepsilon\bar{X}_1(\bar{x}) = \lim_{T\to\infty} T^{-1}\int_{t_0}^{t_0+T} X[\bar{x}, \varphi(\bar{x}, y_0, t_0, t), t, 0]dt.$$

The relationship between the solutions of (1) and (3) is studied over long time-intervals of the order of $1/\varepsilon$.

The principal theorem, which has a long complicated hypothesis, establishes conditions under which $|x-\bar{x}|$ remains arbitrarily small for arbitrarily long time-intervals of the order of $1/\varepsilon$ if ε is sufficiently small. The proof is not given.

The ideas are applied to systems with slowly varying parameters, and to canonical systems $q'=\partial H/\partial p-\varepsilon f^{(p)}(p, q, x, t, \varepsilon)$, $p'=-\partial H/\partial q+\varepsilon f^{(q)}(p, q, x, t, \varepsilon)$, $x'=\varepsilon X(p, q, x, t, \varepsilon)$.

The results are generalizations of the work of Bogolyubov [*O nekotoryh statističeskih metodah v matematičeskoi fizike*, Akad. Nauk Ukrain. SSSR, 1945; MR **8**, 37] and Bogolyubov and Zubarev [Ukrain. Mat. Ž. **7** (1955), 5–17; MR **17**, 217]. Also see Bogolyubov and Mitropol'skiĭ, *Asimptotičeskie metody v teorii nelineĭnyh kolebaniĭ* [2nd ed., Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1958; MR **20** #6812; Chap. V, VI].

W. S. Loud (Minneapolis, Minn.)

9694:

Volosov, V. M. Higher approximations in averaging. Dokl. Akad. Nauk SSSR **137** (1961), 1022–1025 (Russian); translated as Soviet Math. Dokl. **2**, 382–385.

The author continues the study begun in the paper reviewed in #9693. He considers a higher approximation to the averaged system, namely $\dot{\bar{x}}=\varepsilon\bar{X}_1(\bar{x})+\varepsilon^2 A_2(\bar{x})$, $\dot{\bar{y}}=Y_0(\bar{x}, \bar{y}, t)+\varepsilon B_1(\bar{x})$. Here $\bar{X}_1(\bar{x})$ is as in the review above, $Y_0=Y(x, y, t, 0)$, and A_2 and B_1 are defined

appropriately, but by very complicated expressions. The theorems given have extremely long and complicated hypotheses, and proofs are not given. The results are that for the higher approximation, the solutions of the averaged system approximate those of the original system (1) in the above review. Indeed given $\delta>0$ and $K>0$, there exists $\varepsilon_1>0$ such that for $0\leq\varepsilon\leq\varepsilon_1$ and $|t-t_0|<K/\varepsilon$, $|y-\bar{y}|<\delta$ and $|x-\bar{x}-\varepsilon u_1(\bar{x},\bar{y},t)|<\varepsilon\delta$, where u_1 is an explicitly computed improvement to x. Under certain other hypotheses the approximations for y and x are obtained as $O(\varepsilon)$ and $O(\varepsilon^2)$ respectively.

W. S. Loud (Minneapolis, Minn.)

9695:

Myškis, A. D.; Naumovič, A. F. Refinement of the method of reciprocal sequences for the investigation of differential equations with lagging argument. Dokl. Akad. Nauk SSSR **124** (1959), 976–979. (Russian)

Investigation of the solution of the linear differential equation with lagging argument

$$(*)\qquad y''(x) = -M(x)y(x-\Delta(x)),$$

$A\leq x<\infty$, $0\leq M(x)\leq M_0<\infty$, $0\leq\Delta(x)\leq\Delta_0<\infty$, the initial function $y(x)=\varphi(x)$ being given on the interval $[A-\Delta_0, A]$, and all these functions being continuous.

By a modification of the method of recursion relations the following theorems are proved. I. If $y(x)\leq y(A)>0$, $A\leq x\leq B$, $y(B)=0$, then $\max\varphi(x)>y(A)$. II. If $\varphi(x)\geq 0$, then the set of zeros of $y(x)$ for $A\leq x<\infty$ is connected or empty. III. The solution of equation (1) can be of just three types: (a) different from zero for sufficiently large x, (b) changing sign on every interval $[D-\Delta_0, D]$, $A\leq D<\infty$, (c) equal to zero for sufficiently large x. Theorem IV gives estimates for solutions of type (a).

L. È. Èl'sgol'c (RŽMat **1960** #326)

9696:

Norkin, S. B. Periodic motions of a class of oscillatory systems with lagging forces. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **1959**, no. 5, 109–121. (Russian)

Es werden die Bedingungen untersucht, unter denen die Schwingungsgleichung

$$\ddot{x}(t)+2\mu\dot{x}(t)+\nu^2x(t)+T(t)x[t+\Delta(t)] = 0$$

periodische Lösungen besitzt. Dabei ist $T(t)$ periodisch und $\Delta(t)$ eine nicht konstante Zeitverzögerung, die aus dem physikalischen Problem zu bestimmen ist. Die Großzügigkeit, mit der das in der Arbeit behandelte praktische Beispiel—der elektrische Summer—mathematisch formuliert wird, steht in einem auffallenden Gegensatz zu der Sorgfalt, mit der unter Verwendung bekannter Sätze von Myschkis die Existenzbeweise für die gesuchten Lösungen durchgeführt werden. Da keine Deutungen vorgenommen werden, haben die erhaltenen Ergebnisse rein formalen Charakter.

K. Magnus (Stuttgart)

9697:

Šimanov, S. N. On the stability in the critical case of a zero root for systems with time lag. Prikl. Mat. Meh. **24** (1960), 447–457 (Russian); translated as J. Appl. Math. Mech. **24**, 653–668.

Let x denote an n-vector-valued function on $[0, \infty\rangle$. Let x_t be the function defined on $[-\tau, 0]$ by $x_t(\theta)=x(t+\theta)$, $-\tau\leq\theta\leq 0$. Differential equations with lag can then be expressed as an (ordinary) functional differential equation $x_t'(0)=X(x_t)$ where X is a functional defined on a suitable space of functions which are defined on the interval $[-\tau, 0]$ and $x_t'(0)=x'(t)=dx(t)/dt$. Assume that $X=A+R$ where A is a linear functional and R is of higher order and satisfies an analyticity condition. Assume that A has one zero characteristic and the remaining roots are negative and bounded away from zero. A. M. Liapunov showed for ordinary differential equations that the nonlinear terms determine the stability and gave criteria for stability and instability. Results for functional differential equations which are completely analogous to those for ordinary differential equations are obtained in this paper for the critical case of one zero root. The method of study is also analogous to Liapunov's. Several examples illustrate the applicability of the method.

J. P. LaSalle (Baltimore, Md.)

9698:

Qin, Yuan-xun; Liou, Iong-qing; Lian, Wang. Effect of time-lags on stability of dynamical systems. Sci. Sinica **9** (1960), 719–747.

The authors consider the general problem of determining when the stability of the trivial solution of $u'(t)=g(u(t), u(t-\tau))$ can be deduced from that of the comparison equation $u'(t)=g(u(t), u(t))$. A typical result is the following: If $p+q<0$, the trivial solution of $u'(t)=pu(t)+qu(t-\tau)$ is stable provided $0<\tau<\Delta(p, q)=\pi/q(|p|+|q|)$. There are corresponding results for nonlinear equations and for other types of linear and nonlinear differential-difference equations.

R. E. Bellman (Santa Monica, Calif.)

9699:

Ehrmann, Hans. Ein Existenzsatz für die Lösungen gewisser Gleichungen mit Nebenbedingungen bei beschränkter Nichtlinearität. Arch. Rational Mech. Anal. **7** (1961), 349–358.

Work by Conti, this reviewer, and others on existence and continuity of solutions for non-linear, ordinary differential equations together with initial, or boundary, conditions at more than one point is extended to Banach spaces in such way that many of the earlier results occur as special cases. The equation considered is $Lu=Mu$, where L and M are continuous operators, with L being linear. The operators are defined in Banach spaces and subject to sufficient hypotheses to give them the essential properties of the differential systems for which the earlier results were established. Applications are made to certain non-linear difference systems, and to differential systems with non-linear boundary conditions.

W. M. Whyburn (Chapel Hill, N.C.)

9700:

Millionščikov, V. M. A contribution to the theory of differential equations $dx/dt=f(x, t)$ in locally convex spaces. Dokl. Akad. Nauk SSSR **131** (1960), 510–513 (Russian); translated as Soviet Math. Dokl. **1**, 288–291.

The author generalizes the standard existence and uniqueness theorems for the equation (1) $\dot{x}=f(x, t)$, to the case where x is an element of a complete locally convex space. He first gives a definition of an integral on such a space, and lists its properties. Using the contraction-mapping principle, the existence and uniqueness theorem is proved, assuming that $f(x, t)$ is Lipschitzian in x with

respect to each pseudo-norm, with the same Lipschitz function for all pseudo-norms. Then conditions on f are given, so that Tychonov's theorem may be applied, and several theorems concerning stability of (1) are proved by comparison of (1) with a certain scalar differential equation. *A. Stokes* (Baltimore, Md.)

PARTIAL DIFFERENTIAL EQUATIONS

See also 9581, 9843.

9701:

Sagan, Hans. ★Boundary and eigenvalue problems in mathematical physics. John Wiley & Sons, Inc., New York-London, 1961. xviii+381 pp. $9.50.

The topics covered, by chapters, are as follows: (I) Hamilton's principle and the theory of the first variation. This includes several independent variables, the isoperimetric problem and natural boundary conditions. (II) Representation of some physical phenomena by partial differential equations. This includes the vibrating string and membrane, heat conduction and the potential equation. (III) Theorems related to partial differential equations and their solutions. This includes a discussion of existence and uniqueness, the problems of minimal surface, Cauchy-Kowalewski and the separation method. (IV) Fourier series. This deals with the formal properties, the convergence proof for sectionally smooth functions and some applications to partial differential equations. (V) Self-adjoint boundary value problems. This includes second order ordinary differential equations, Sturm-Liouville problems, eigenvalues, oscillations, orthogonality. Completeness and expansion theorems are discussed, but proofs are deferred to the references. (VI) Legendre polynomials and Bessel functions. (VII) Characterization of eigenvalues by a variational principle. This includes the minimum properties of the eigenvalues of a self-adjoint boundary value problem, the method of Rayleigh and Ritz. (VIII) Spherical harmonics. This includes applications to Schrödinger's equation for the hydrogen atom, Poisson's integral representation of the solution of the potential equation. (IX) The nonhomogeneous boundary value problem. This deals with Green's functions and their application. An appendix reviews Vector analysis, Convergence and Ordinary differential equations.

The student who has had a standard course in advanced calculus should have no trouble reading this text. The presentation is clear and careful and the historical remarks are often entertaining. The language of abstract vector spaces is avoided; Hilbert or Banach spaces do not appear. Nor is any coverage given to finite difference methods or numerical approximation.

H. D. Block (Ithaca, N.Y.)

9702:

Žukov, A. I. On the problem of convergence of difference methods of solution of the Cauchy problem. Vyčisl. Mat. **6** (1960), 34–62. (Russian)

The question of solving Cauchy's problems for partial differential equations by difference methods is considered, when conditions for stability of the difference scheme (the so-called Courant's conditions) are not fulfilled. The author begins with a simple example of the parabolic equation

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} \tag{1}$$

with the well known difference scheme

$$u(x, t+\tau) = u(x, t)+a\tau h^{-2}[u(x-h, t)-2u(x, t)+u(x+h, t)], \tag{2}$$

which is stable if $0 \leqq a\tau/h^2 \leqq 1/2$. Choosing the initial condition

$$u(x, 0) = \begin{cases} (1-x^2)^2, & |x| \leqq 1, \\ 0, & |x| > 1, \end{cases}$$

he presents a graph of the solution for $t=0.05$, when the difference scheme (2) is unstable: $a=1$, $h=0.1$, $\tau=0.01$, so that $a\tau/h^2=1$.

It is easily seen from the graph that the solution has an oscillating character; but the oscillations are equally distributed along the curve of the exact solution. A question arises, whether the difference method solution, represented by the oscillating curve, could not be interpreted as an approximation of the exact solution (for $t=0.05$) in the sense of a certain functional space. In fact, such a space exists (not only for the case of the simple example treated above, of course, but for a rather general class of problems); it is the space of the so-called generalised functions, construction and investigation of which make the substance of the work.

Finally, the author solves a practically important question of "smoothing" of these oscillating solutions and presents some useful examples. *K. Rektorys* (Prague)

9703:

Kunin, I. A. Approximative method for solution of boundary problems for certain equations of elliptic type. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1958**, no. 10, 146–150. (Russian)

Let Ω be a region in n-dimensional space and the closed bounded surface S its boundary. Suppose that $\nu(x)>0$ and is continuous in Ω. A method of solving $\nabla(\nu\nabla p)=-\rho$ with $p=0$ on S is proposed. The author states that νp can be expressed formally by a series of functions satisfying certain Poisson equations and that the series converges rapidly, with the first term giving a satisfactory approximation in some cases, provided $\|\nabla\nu\|$ is small. He shows that if one can find a particular solution ξ of $\nabla(\nu\nabla\xi)=0$ such that $\|\nabla(\xi^2\nu)\|$ is small in Ω and if the equation $\nabla(\nu^0\nabla p^0)=-\rho^0$, where $\nu^0=\xi^2\nu$ and $\rho^0=\xi\rho$, is solved by the method of constructing the series given above, then $p=\xi p^0$ is the required solution.

The method is applied to a differential equation describing the pressure distribution in the lubricant of a bearing in the form of a sector of a cylinder.

Č. Masaitis (Aberdeen, Md.)

9704:

Kuznecov, N. N. Problem of the disintegration of an arbitrary discontinuity for a system of quasilinear equations of the first order. Dokl. Akad. Nauk SSSR **131** (1960), 503–506 (Russian); translated as Soviet Math. Dokl. 1, 282–285.

Hyperbolic systems of the form $u_t+(\varphi_i(u))_x=0$ $(i=1, 2)$

with some additional restrictions on φ are considered. A proof is sketched that initial-value problems of the form $u(0,x)=u^+=\text{const}$ for $x>0$, $u(0,x)=u^-=\text{const}$ for $x<0$ has one and only one solution of the form $u=u(x/t)$ if only "stable" discontinuities are permitted, as is customary. *P. Ungar* (New York)

9705:

Danilyuk, I. I. On the general representation of the solutions of an axial-symmetric stationary problem. Dokl. Akad. Nauk SSSR **132** (1960), 743–746 (Russian); translated as Soviet Math. Dokl. **1**, 626–629.

Using a formula due to I. N. Vekua [*Novye metody rešeniya ellipličeskih uravnenii*, OGIZ, Moscow-Leningrad, 1948; MR **11**, 598] of the general representation of the solutions of the equation (1) $F_{z\zeta}+AF_z+CF_\zeta=0$, where $F_z=\partial F/\partial z$, etc., the author establishes the general representation of regular solutions of the equation $F_\zeta+AF+BF^*=0$, where $F^*=\overline{F(\zeta,z)}$ $(\zeta=\bar{z})$, $A(z,\zeta)$ and $B(z,\zeta)$ are regular in a certain bicylindrical region $D(z)\times D(\zeta)$, $D(\bar{z})=D(\zeta)$, and satisfy the conditions $A_z-BB^*=-CA$ and $B_z-BA^*=-CB$ with a certain regular C. It involves the Riemann function of (1), B, C and an arbitrary regular function. In the case of the steady, incompressible, rotationally symmetrical flow, where $A=B=-1/2(z-\zeta)$, the Riemann function is expressed in terms of the second complete elliptic integral. Finally, the author mentions the possibility of solving the steady incompressible flow around a body of revolution by a method of singular integrals over the contour. *S. S. Shu* (Lafayette, Ind.)

9706:

Maslennikova, V. N. Mixed problems for a system of partial differential equations of first order. Izv. Akad. Nauk SSSR Ser. Mat. **22** (1958), 271–278. (Russian)

The system $\partial\mathbf{v}/\partial t-\mathbf{v}\times\mathbf{k}+\text{grad}\,p=\mathbf{F}$, $\partial p/\partial t+\text{div}\,\mathbf{v}=\psi$, with initial conditions $\mathbf{v}|_{t=0}=\mathbf{v}^0(x)$, $p|_{t=0}=p^0(x)$, is solved with either of the two boundary conditions $p|_S=0$ or $v_n|_S=0$, where S is a surface in $x_1x_2x_3$-space bounding a domain Ω in which the system is defined. At the same time, related boundary-value problems for the equation $\partial^2\Delta u/\partial t^2+\partial^2 u/\partial x_3^2-\partial^2 u/\partial t^2-\partial^4 u/\partial t^4=f$ are also solved. In all cases conditions are determined under which classical and weak solutions exist and are unique. Continuous dependence of the solutions on the initial data and on the right members is proved, and estimates for the derivatives are obtained. This paper constitutes the complete treatment of what was summarized in an earlier report [Dokl. Akad. Nauk SSSR **102** (1955), 885–888; MR **17**, 631]. *R. N. Goss* (San Diego, Calif.)

9707:

Šabat, B. V. On the notion of derivative system according to M. A. Lavrent'ev. Dokl. Akad. Nauk SSSR **136** (1961), 1298–1301 (Russian); translated as Soviet Math. Dokl. **2**, 202–205.

Following M. A. Lavrent'ev the author considers the derived (quasi-linear) system of partial differential equations associated with a general first-order equation. He presents a new proof and a sharpening of a theorem of Lavrent'ev giving necessary and sufficient conditions for the derived system to be elliptic, and he gives an application. See M. A. Lavrent'ev [Mat. Sb. (N.S.) **21** (**63**) (1947), 285–320; Izv. Akad. Nauk SSSR Ser. Mat. **12** (1948), 513–554; MR **10**, 290; **11**, 650], also M. A. Lavrent'ev and B. V. Šabat [Dokl. Akad. Nauk SSSR **112** (1957), 810–811; MR **19**, 286]. *A. L. Shields* (New York)

9708:

Šilov, G. E. Local properties of solutions of partial differential equations with constant coefficients. Uspehi Mat. Nauk **14** (1959), no. 5 (89), 3–44. (Russian)

A differential operator $P(D)=P(\partial/i\partial x)$ with constant coefficients is said to be hypoelliptic if $Pu=0$, u a distribution, implies that (1) $u\in C^\infty$. As was first shown by Hörmander [Acta Math. **94** (1955), 161–248; MR **17**, 853] P is hypoelliptic if and only if (2) $P(\xi)=0\Rightarrow|\text{Im}\,\xi|\geq c|\text{Re}\,\xi|^h-C$ for some positive constants c and h. This paper is an exposition (with proofs) of this and similar results, e.g., that if (1) is replaced by the requirement that the derivatives of u of order q are locally bounded by $C^{\nu+1}q^{\beta q}$ $(\beta\geq 1$; C a constant depending on $u)$ then (2) holds with $h=\beta^{-1}$ and conversely. *L. Gårding* (Lund)

9709:

Collins, W. D. On the solution of some axisymmetric boundary value problems by means of integral equations. III. Some electrostatic and hydrodynamic problems for two spherical caps. Proc. London Math. Soc. (3) **10** (1960), 428–460.

In two recent papers [Quart. J. Mech. Appl. Math. **12** (1959), 232–241; Mathematika **6** (1959), 120–133; MR **21** #4743; **22** #7644], the author has given a method of solving potential problems for a circular disc or spherical cap. He now extends the method to deal with potential and related problems in electrostatics and perfect fluid theory for two or more caps or discs which have the same axis of symmetry.

The method starts by expressing the potential function or stream function as the real or imaginary part of a certain complex integral, and the problem is reduced to the solution of a set of Volterra integral equations (in one independent variable only) of the first kind followed by the solution of a set of Fredholm equations of the second kind.

The problems solved include (i) the determination of the potential due to two perfectly conducting caps maintained at given potentials symmetrical about the axis of symmetry; (ii) the special case of (i) for two spheres; (iii) the special case of (i) when the caps intersect and form a lens-shaped conductor; (iv) the limiting case of (i) for a pair of discs.

The procedure can be extended to any number of caps or discs, and this enables the author to solve the potential problem for a disc between two parallel conducting planes. *E. T. Copson* (St. Andrews)

9710:

Kapilevič, M. B. On connection formulas for singular Tricomi problems. Dokl. Akad. Nauk SSSR **132** (1960), 28–31 (Russian); translated as Soviet Math. Dokl. **1**, 466–469.

The singular Tricomi problems considered are to find

the solutions $u(x, y, \beta)$, $\bar{u}(x, y, \beta)$ of the Euler-Poisson equation

$$(1) \quad (y-x)u_{xy}+\beta(u_x-u_y) = 0 \qquad (0 \leq a = 2\beta < 1)$$

which belong to class C^2 in the region $D(y>x>0)$ and which satisfy the conditions $u(x, x)=f(x)$, $u(0, y)=0$, or $\bar{u}_\eta(x, x)=f(x)$, $\bar{u}(0, y)=0$ $(\eta=-[(y-x)/(2-2a)]^{1-a})$ respectively on the rays $y=x\geq 0$, $x=0$, $y\geq 0$; $f\in C^2$ on $y=0$, $x\geq 0$ and $f(0)=0$. Connection formulas are stated which express $u(x, y, \beta_2)$ in terms of $u(x, y, \beta_1)$ and $\bar{u}_x(x, y, \beta_1)$ and similarly for $\bar{u}(x, y, \beta_1)$. An example of these formulas is

$$(2) \quad u(x, y, \beta_2) = (y-x)^{1-\beta_1-\beta_2}\int_0^x K_1(x, y, \xi, \beta_1, \beta_2)u(\xi, y, \beta_1)\,d\xi,$$

where $\beta_2>\beta_1\geq 0$, $\beta=\beta_2-\beta_1$,

$$K_1 = \kappa_1(y-\xi)^{\beta_1-1}(x-\xi)^{\beta-1}F(-\beta, \beta_2, \beta; \omega),$$

$\omega(x-y)=x-\xi$ and κ_1 is a constant.

Analogous formulas are obtained for u and the solutions $z(x, y, \beta)$ and $\bar{z}(x, y, \beta)$ of two singular Goursat problems:

$$(3) \quad yz_{xy}+\alpha z_y+\beta z_x = 0 \qquad (\alpha > 0, \beta > 0, (x, y)\in D)$$

(a) $z(0, y)=0$, $z(x, 0)=f(x)$, or (b) $\bar{z}(0, y)=0$, $\bar{z}_y(x, 0)=f(x)$, as well as between u and the solutions of other singular boundary value problems.

Now let the fixed initial function f be replaced by $f_2=Pf_1$, where P is an arbitrary function of x, integrable on any finite interval of $y=0$, $x\geq 0$. Some of the connection formulas also hold in this case. Consideration is given to the cases $P(x)=(x-x_1)^{-k_1}\cdots(x-x_n)^{-k_n}$ and $P(x)=\exp(kx)$. *J. Mitchell* (University Park, Pa.)

9711:

Kapilevič, M. B. The problem of mixed boundary conditions for singular hyperbolic equations. Dokl. Akad. Nauk SSSR **132** (1960), 1005–1008 (Russian); translated as Soviet Math. Dokl. **1**, 693–696.

Solutions $u(x, y, \beta, \beta')$ and $\bar{u}(x, y, \beta, \beta')$ of the equation

$$(1) \quad (y-x)u_{xy}+\beta' u_x-\beta u_y = 0 \qquad (0 \leq \beta, \beta' < \tfrac{1}{2})$$

are considered, which $\in L_2$ in the region $D(0\leq x\leq y\leq x_0)$ of the half-plane $y\geq x$ and which satisfy the boundary conditions (a) $u(0, y)=0$, $u(x, x)=\tau(x)$, $\tau(0)=0$, (b) $\bar{u}(0, y)=0$, $\bar{u}_\eta(x, x)=\nu(x)$, respectively; τ and $\nu\in C^2$ on $(0, x_0)$ and $\eta=-[(y-x)/(2-a-a')]^{1-\beta-\beta'}$ $(a=2\beta, a'=2\beta')$. If $\tau_2=P\tau_1$ where P is an arbitrary function of x, integrable on $(0, \bar{x}_0)$ $(0<\bar{x}_0\leq x_0)$, a connection formula is given for the corresponding solutions $u_1=u(x, y, \beta_1, \beta_1')$ and $u_2=u(x, y, \beta_2, \beta_2')$ which satisfy (a). [For an example of a connection formula see (2) of #9710.] In particular if $\beta_2=\beta_1$ and $\tau=\tau_1=\tau_2$ and $\tau\in C^{N+1}$ in $(0, x_0)$ the connection formula can be replaced by an expansion with remainder term. If $\tau\in C^\infty$, the expansion becomes an infinite double series which reduces symbolically to the Humbert function. Analogous results are given for the solutions $\bar{u}$. If $\tau=P\nu$, similar results hold for $\bar{u}_1$ and u_2. The confluent case of (1) is obtained by replacing x by εx in (1) and $\beta=-\alpha/\varepsilon$ in (a) and letting $\varepsilon\to 0$, giving the singular Goursat problem of (3) and (3a) in #9710. Relations are given between z_1 and z_2 and u_2 and z_1.

J. Mitchell (University Park, Pa.)

9712:

Čibrikova, L. I. Application of automorphic functions to the solution of some boundary problems for equations of mixed type. Issledovaniya po sovremennym problemam teorii funkciĭ kompleksnogo peremennogo, pp. 515–519. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960. (Russian)

It is shown that an explicit solution of the Tricomi problem can be given when the upper part of the boundary forms half of the boundary of a fundamental region for a suitable group of bilinear transformations of the complex plane. The author's results, based on her previous paper in Kazan. Gos. Univ. Uč. Zap. **116** (1956), no. 4, 59–110, coincide in the special case of a semi-circle with one due to A. V. Bicadze [Trudy Mat. Inst. Steklov. **41** (1953); MR **16**, 43]. *F. V. Atkinson* (Toronto)

9713:

Lanczos, C. Extended boundary value problems. Proc. Internat. Congress Math. 1958, pp. 154–181. Cambridge Univ. Press, New York, 1960.

The author's goal in this paper is to establish a common platform for the theory of linear partial differential equations subject to boundary conditions which may or may not be chosen judiciously and which may be too many or too few. This problem is approached by first solving the general problem for a system of algebraic equations on the basis of an eigenvalue problem which yields a unique solution if the proper compatibility conditions are satisfied. The method is then translated into the domain of differential equations. The general boundary value problem belongs to one of two categories depending on whether or not the spectrum of the operator has zero as a limit point. The conventional boundary value problems are those of the first category, namely those for which zero is not a limit point of the spectrum. The eigenvalue method is, however, applicable to either category. The solvability of the problem depends on whether the data satisfy certain compatibility relations (and in the second category certain convergence conditions) which can be explicitly stated. The explicit solution and the testing of the data concerning solvability hinges, however, on the preliminary determination of the solution of the eigenvalue problem.

L. E. Payne (Newcastle upon Tyne)

9714:

Il'in, V. A.; Šišmarev, I. A. The connection between generalized and classical solutions of the Dirichlet problem. Izv. Akad. Nauk SSSR. Ser. Mat. **24** (1960), 521–530. (Russian)

Les auteurs considerent l'équation élliptique autoadjointe $Lu=-f$, avec la condition de Dirichlet $u|_\Gamma=0$. Ils démontrent que dans les conditions d'existence et unicité pour la solution classique, celle-ci réalise le minimum de l'intégrale de Dirichlet dans le complété par rapport à la norme de $W_2^{(1)}$ (notation de S. L. Sobolev) de l'espace des fonctions continues avec leurs dérivées de premier ordre et ayant les supports inclus dans l'intérieure du domaine considéré. *G. Marinescu* (Bucharest)

9715:

Boyarskiĭ, B. V. On the Dirichlet problem for a system of elliptic equations in space. Bull. Acad. Polon. Sci. Sér.

Sci. Math. Astr. Phys. **8** (1960), 19–23. (Russian. English summary)

Author's summary: "Using a result by Lopatinskiĭ the author proves that the Dirichlet problem for systems of two second-order elliptic equations in n-dimensional space, $n \geq 3$, is always equivalent to a system of Fredholm integral equations. Topological reasons for and generalizations of this fact are given." *R. Finn* (Stanford, Calif.)

9716:

Bakel'man, I. Ya. The Dirichlet problem for equations of Monge-Ampère type and their n-dimensional analogues. Dokl. Akad. Nauk SSSR **126** (1959), 923–926. (Russian)

The author extends some of the results of his earlier work [same Dokl. **114** (1957), 1143–1145; MR **20** #1983] to n-dimensional analogue of the Monge-Ampère equation of the following form:

$$\Gamma(z) - \sum_{i,k=1}^{n} A \frac{\partial^2 z}{\partial x_i \partial x_k} - E = 0$$

where $\Gamma(z)$ is the Hessian of $z(x_1, \cdots, x_n)$, $n \geq 2$, A_{ik} and E are functions continuous with respect to the variables $x_1, \cdots, x_n$, z, $p_1, \cdots, p_n$ ($p_i = \partial z/\partial x_i$) in an open n-dimensional domain D bounded by $(n-1)$-dimensional convex surface Γ, and continuous with respect to z, $p_1, \cdots, p_n$ on $D+\Gamma$. Imposing the conditions

$$\sum_{i,k=1}^{n} A_{ik}\xi_i\xi_k \geq 0$$

for all real ξ_i, and $0 \leq E \leq \varphi(x_1, \cdots, x_n)R(p_1, \cdots, p_n)$, where φ is a non-negative summable function over D and $R \geq R_0 = \text{const} > 0$ is continuous for all $p_1, \cdots, p_n$, the author determines sufficient conditions for the existence (generally not unique) of a generalized solution (in the sense of Gelfand and Šilov) of the corresponding Dirichlet problem in the class $W(D)$ of convex functions over D [cf. same author, Vestnik Leningrad. Univ. Ser. Mat. Meh. Astr. **13** (1958), no. 1, 25–38; MR **20** #3384].

V. Linis (Ottawa, Ont.)

9717:

Vaĭnberg, B. R. The asymptotic behavior of Green's function for the Sobolev-Gal'pern equation. Dokl. Akad. Nauk SSSR **136** (1961), 1015–1018 (Russian); translated as Soviet Math. Dokl. **2**, 147–151.

The author finds asymptotic formulas for Green's functions for the Cauchy problem for equations of the form $P(\partial/\partial t, i\partial/\partial x)u = 0$, for which $\operatorname{Re} \lambda_j(\sigma) < c$, $j = 1, 2, \cdots, l$; $P(\lambda, s)$ is a polynomial with constant coefficients; $s = \sigma + i\tau$; and $\lambda_j(s)$ are the roots of $P(\lambda, s) = 0$. These asymptotic expressions preserve differentiation. The author then investigates classes of solutions of the Cauchy problem for these equations.

S. Hoffman (Hartford, Conn.)

9718:

Lax, P. D.; Phillips, R. S. Local boundary conditions for dissipative symmetric linear differential operators. Comm. Pure Appl. Math. **13** (1960), 427–455.

This paper treats the linear differential operator $L = \sum_{j=1}^{n} A^j \partial/\partial x^j + B$, where A^j and B are $n \times n$ matrix-valued functions defined on a smooth bounded domain G contained in R_n. The operator acts on vector-valued functions u defined on G and satisfying the boundary condition that $u(x) \in N(x)$ for all x on the boundary S of G, $N(x)$ being a smoothly varying linear subspace of constant dimension. Let $n(x) = (n_1, \cdots, n_n)$ be the outer normal to G at $x \in S$, and set $A_n = \sum n_j A^j$. It is shown that if A_n is non-singular everywhere on S, then any weak solution to $Lu = f$ is a strong solution. This is done using various L_2 spaces and the notion of mollifiers. [Compare K. O. Friedrichs, Trans. Amer. Math. Soc. **75** (1944), 132–151; Comm. Pure Appl. Math. **11** (1958), 333–418; MR **5**, 188; **20** #7147.] It is shown also that if A_n is merely of constant rank, then any weak solution is what the authors call "semi-strong".

The operator L is said to be dissipative and symmetric if the A^j are all symmetric, if $B + B^* - \sum A^j_{x^j}$ is non-positive, and if $uA_n u \leq 0$ for all $u \in N(x)$, $x \in S$. In this case it is shown that $u - Lu = f$ always has a unique strong solution for each square-integrable function f. This result is further extended to cases where S contains corners of suitable type and where the A^j are suitably restricted. As an application, particularly of these latter results, the authors extend the work of Morawetz [ibid. 315–331; MR **20** #3375] and show that the Frankl problem has a unique strong solution. *G. Hufford* (Seattle, Wash.)

9719:

Cordes, H. O. On maximal first order partial differential operators. Amer. J. Math. **82** (1960), 63–91.

This paper is concerned with the boundary value problem for the differential operator

$$L_1 u = \sum_{1}^{n} a_i(x)\partial u/\partial x_i + b(x)u,$$

where $a_i(x)$ and $b(x)$ are $m \times m$ matrix-valued functions continuous on a smooth domain D in real Euclidean n-space and on its boundary Γ. The matrices $a_i(x)$ have Hölder-continuous first derivatives on $D+\Gamma$. It is further assumed that iL_1 is formally self-adjoint, that is, that $b(x) + b^*(x) = \sum_1^n \partial a_i(x)/\partial x_i$. The operator L_1 acts on the Hilbert space of vector-valued functions $u(x)$, $v(x)$ with inner product $(u, v) = \int_D \langle u, v \rangle dx$, $\langle u, v \rangle$ being the ordinary m-dimensional Euclidean inner product. For smooth functions $u(x)$ it is readily seen that $(u, Lu) + (Lu, u) = \int_\Gamma \langle Au, u \rangle d\sigma$ where $A = \sum_1^n \nu^i a_i$ and $\{\nu^i(x)\}$ denotes the outer normal to Γ at x. The problem is to characterize boundary conditions so that L_1 restricted to smooth functions satisfying the given conditions on Γ defines a maximal dissipative operator L, that is, $(u, Lu) + (Lu, u) \leq 0$ and L is maximal with respect to this property. This problem has previously been considered in an abstract way by the reviewer [Trans. Amer. Mat. Soc. **90** (1959), 193–254; MR **21** #3669], and concretely for local boundary conditions by K. O. Friedrichs [Comm. Pure Appl. Math. **11** (1958), 333–418; MR **20** #7147] and P. D. Lax and R. S. Phillips [above, #9718]. The author, on the other hand, is able to treat a class of non-local, as well as local, boundary conditions in a concrete fashion. Setting $B(x)$ equal to the positive square root of $(A(x))^2$, he defines an auxiliary boundary space E of vector-valued functions on Γ with inner product $[u, v] = \int_\Gamma \langle Bu, v \rangle d\sigma$. A subspace N of E is called negative if $\int_\Gamma \langle Au, u \rangle d\sigma \leq 0$ for all u in N. Not all maximal negative subspaces define boundary conditions for maximal dissipative restrictions of L_1. However, letting $P_+ = \frac{1}{2}(B + A)$ and $P_- = \frac{1}{2}(B - A)$, a contraction operator J on

P_-E to P_+E can be defined as $JP_-u=P_+u$ for u in N. The author shows that for those maximal negative subspaces N for which the norm of J is less than one, the corresponding restriction of L_1 is essentially maximal dissipative. The main result of the paper generalizes somewhat the previous assertion.

R. S. Phillips (Palo Alto, Calif.)

9720:

Berezanskiĭ, Yu. M. Some examples of nonclassical boundary-value problems for partial differential equations. Dokl. Akad. Nauk SSSR **131** (1960), 478–481 (Russian); translated as Soviet Math. Dokl. **1**, 259–262.

The boundary-value problem $\mathscr{L}[u]=f\in H^{-,*}$, $u\in(b)$, is correct and uniquely solvable in the weak sense provided that the inequalities

$$\|\mathscr{L}^+[v]\|_0 \geq c\|v\|_{+,*} \quad (c>0,\ v\in W_2^r(b)^+),$$
$$\|\mathscr{L}[u]\|_0 \geq c\|u\|_+ \quad (u\in W_2^r(b)),$$

are satisfied. Here $\mathscr{L}$ is a linear differential operator of rth order in a bounded region G of E_n with complex coefficients, $\mathscr{L}^+$ is its formal adjoint, $W_2^r(b)$ is a certain subspace of a Sobolev space W_2^r comprising those functions which satisfy given boundary conditions (b), $H^{-,*}$ is a Hilbert space with negative norm $\|\ \|_{-,*}$, and the other norms are defined analogously. In this paper these inequalities are derived as special cases of Hörmander's inequality $\|\mathscr{L}[u]\|_0\geq c\|\mathscr{L}^{(\sigma)}[u]\|_0$ [Acta Math. **94** (1955), 161–248; MR **17**, 853], when the latter is extended to certain classes of functions u which do not remain finite in G. This extension is effected by an integration by parts of the integral $\int_G x_k\mathscr{L}[u]\overline{\mathscr{L}^{(k)}[u]}\,dx$. $\mathscr{L}^{(k)}[u]$ is a differential expression formed from $\mathscr{L}[u]$ by operating with D^k in the Schwartz notation. The method is used to establish the existence of weak solutions for several special boundary-value problems that do not fall under the classical canonical types.

R. N. Goss (San Diego, Calif.)

9721:

Il'in, V. A. On solvability of mixed problems for hyperbolic and parabolic equations. Uspehi Mat. Nauk **15** (1960), no. 2 (92), 97–154 (Russian); translated as Russian Math. Surveys **15** (1960), no. 2, 85–142.

The author presents an extensive work concerning solvability of hyperbolic and parabolic boundary-value problems for a very general type of region, the so-called normal region [resp. corresponding normal cylinder]. A region is called normal if Dirichlet's problem for Laplace's equation is solvable for this region for any continuous boundary function. Two problems are investigated, the hyperbolic and the parabolic one:

$$Lu-u_{tt}=-f(x,t),$$
$$u(x,0)=\varphi(x),\quad u_t(x,0)=\psi(x),\quad u|_{x\in\Gamma}=0;$$

and

$$Lu-u_t=-f(x,t),$$
$$u(x,0)=\varphi(x),\quad u|_{x\in\Gamma}=0.$$

Here $x\in g$ (g is an N-dimensional region with boundary Γ), $0\leq t\leq l$,

$$Lu=\sum_{i,j=1}^{N}\frac{\partial}{\partial x_i}\left[a_{ij}(x)\frac{\partial u}{\partial x_j}\right]-c(x)u,$$

$a_{ij}=a_{ji}$, and $\sum_{i,j=1}^{N}a_{ij}(x)\xi_i\xi_j\geq\alpha\sum_{i=1}^{N}\xi_i^2$ for any real vector $(\xi_1,\cdots,\xi_N)$, $\alpha>0$.

The author presents first an extensive survey of the literature touching similar problems, specifically, as to properties of the region g which are required in these works. Particular attention is paid to work of O. A. Ladyženskaya. The fundamental purpose of the present paper is to prove that the problems mentioned above are solvable even in the classical sense (under some not too restrictive suppositions about the smoothness of the functions a_{ij}, c, φ, ψ, which are profoundly analysed later) for any normal region g. Since this is a well-known result for corresponding elliptic problems [see, e.g., Oleĭnik, Mat. Sb. (N.S.) **24** (**66**) (1949), 3–14; Tautz, Math. Nachr. **2** (1949), 279–303; Keldyš, Uspehi Mat. Nauk **8** (1941), 171–231; MR **10**, 713; **11**, 358; **3**, 123], the author's work shows a uniform criterion as to the regions investigated, for solvability of equations of all the three types.

The proof is based on the use of the Fourier method and on an extensive study of the properties of the corresponding eigenfunctions.

K. Rektorys (Prague)

9722:

Krylov, V. Yu. Some properties of the distribution corresponding to the equation $\partial u/\partial t=(-1)^{q+1}\partial^{2q}u/\partial x^{2q}$. Dokl. Akad. Nauk SSSR **132** (1960), 1254–1257 (Russian); translated as Soviet Math. Dokl. **1**, 760–763.

Following a suggestion of Gelfand, the author obtains a Feynman-Kac type formula for the solution to the Cauchy problem

$$\frac{\partial u}{\partial t}(x,t)=(-1)^{q+1}\frac{\partial^{2q}u}{\partial x^{2q}}(x,t)-V(x)u(x,t),$$
$$u(0+,x)=\psi(x).$$

Namely, the solution is obtained as

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\cdots\int\psi(x_n)\times\exp\{-tn^{-1}\sum_{k=0}^{n-1}V(\chi_k)\}P_{n,x}(dx_1,\cdots,dx_n),$$

where $P_{n,x}$ is a certain (real) measure in n-space which formally generalizes the joint distribution of Brownian motion starting from x and evaluated at the times $t_1,\cdots,t_n$, $t_k=kt/n$ (the case $q=1$). If the $P_{n,x}$ are indexed by the set of time points t/n, $2t/n$, $\cdots$, t, then the usual consistency conditions are satisfied when $(t/n, 2t/n, \cdots, t)\subset(t/m, 2t/m, \cdots, t)$, so the author speaks of a generalized measure P_{2q} (depending also on x). The "measure" has infinite total variation, i.e., tot. var. $P_{n,x}\uparrow\infty$ as $n\to\infty$. An "arcsine law" is given: let $S_r=\{x\in\mathscr{C}(0,t)|x(s)<0$ more than $t-r$ of the time$\}$. Then the measure of S_r (in the appropriate limiting sense) is $2\pi^{-1}\arcsin(r/t)^{1/2}$.

J. Feldman (Princeton, N.J.)

9723:

Krasnov, M. L. Mixed boundary problems for degenerate linear hyperbolic differential equations of second order. Mat. Sb. (N.S.) **49** (**91**) (1959), 29–84. (Russian)

This paper is concerned with problems of existence,

uniqueness and differentiability of generalized solutions of equations of the form

$$Lu \equiv \frac{\partial^2 u}{\partial t^2} - \sum \frac{\partial}{\partial x_i}\left(a_{ik}(x,t)\frac{\partial u}{\partial x_k}\right) + \sum b_i(x,t)\frac{\partial u}{\partial x_i} + e(x,t)\frac{\partial u}{\partial t} + d(x,t)u = h(x,t) \tag{1}$$

where $x=(x_1, \cdots, x_m)$, and $a_{ik}(x,t)=a_{ki}(x,t)$. The first problem treated assumes $\sum a_{ik}(x,t)\xi_i\xi_k \geqq C^2 t^\alpha |\xi|^2$, where $C=$ constant, $\alpha>0$, and seeks a solution in the cylindrical domain $Q=D\times(0<t<l)$, D a domain in E^m with boundary Γ, and subject to the mixed boundary conditions $u|_{t=0}=\partial u/\partial t|_{t=0}=0$ and $u=0$ on $\Gamma\times(0<t<l)$. Existence of a generalized solution is established by means of a modification of Galerkin's method [M. I. Višik, Mat. Sb. (N.S.) **39** (81) (1956), 51–148; MR **18**, 215]. Assuming

$$\partial(\sum a_{ik}t^{-\alpha-\delta_0}\cdot\xi_i\xi_k)/\partial t < 0,$$

a generalized solution is shown to exist satisfying the inequality

$$\iint_Q \left[\frac{1}{t^{\alpha+\delta_0+1}}\left(\frac{\partial u}{\partial t}\right)^2 + \frac{1}{t^{\delta_0+1}}\sum\left(\frac{\partial u}{\partial x_i}\right)^2\right] dxdt < C\iint_Q \frac{h^2(x,t)}{t^{\alpha+\delta_0-1}}\,dxdt.$$

Uniqueness is established within an even broader class of functions.

If in (1) the function $e(x,t)$ is set equal to a/t where a is a constant, and $\sum a_{ik}\xi_i\xi_k \geqq C^2|\xi|^2$, i.e., $\alpha=0$, the mixed boundary problem for the resulting Euler-Poisson-Darboux equation is shown to have a unique solution in the class of functions u such that

$$\iint_Q (u_t{}^2 + a/t + \sum (u_i)^2)\,dxdt < \infty.$$

In the last part of the paper the author studies the case in which the quadratic form becomes degenerate on part of $\Gamma\times(0<t<l)$. Here, boundary conditions the same as above at $t=0$, i.e., $u=\partial u/\partial t=0$, but with $u=0$ on only part of $\Gamma\times(0<t<l)$ and free on the remainder, are shown to uniquely determine solutions under suitable assumptions on the way the quadratic form becomes degenerate on the boundary. Throughout the paper existence is established by a modification of the method of Galerkin. Uniqueness follows from various energy-type inequalities. Interior regularity of the solution, when proved, is based on theorems of Sobolev. This paper extends and sharpens results in previous papers of the author [Dokl. Akad. Nauk SSSR **107** (1956), 789–792; MR **19**, 748; Trudy Energ. Inst. **28** (1956), 25–45].

A. N. Milgram (Minneapolis, Minn.)

9724:

Guglielmino, Francesco. Sul problema di Goursat. Ricerche Mat. **9** (1960), 91–105.

Siano $\alpha(x)$, $\varphi(x)$ funzioni di classe C^1 in $[a,b]$, e siano $\beta(y)$, $\psi(y)$ funzioni di classe C^1 in $[c,d]$, tali che $\alpha(0)=\beta(0)=0$, $\varphi(0)=\psi(0)$, $c\leqq\alpha(x)\leqq d$, $a\leqq\beta(y)\leqq b$. Data una funzione $f(x,y,z,p,q)$, continua e limitata nell'insieme S: $x\in[a,b]$, $y\in[c,d]$, $|z|$, $|p|$, $|q|<+\infty$, si ricercano funzioni $z(x,y)$ nel rettangolo R: $a\leqq x\leqq b$, $c\leqq y\leqq d$, tali che

$$z_{xy}=f(x,y,z,z_x,z_y),\quad z(x,\alpha(x))=\varphi(x),\quad z(\beta(y),y)=\psi(y)$$

(Problema di Goursat).

L'autore, valendosi dei metodi dell'analisi funzionale, prova (Teorema I) che il problema ammette in R almeno una soluzione continua insieme con le derivate z_x, z_y, z_{xy} se esistono due funzioni $L_1(x,y)$, $L_2(x,y)$, entrambe $\geqq 0$ in R, con L_1 sommabile in $y\in[c,d]$, ed L_2 sommabile in $x\in[a,b]$, tali che

$$\sup_{[a,b]}\left\{\int_c^d L_1(x,\eta)\,d\eta + \int_a^b L_2(\xi,\alpha(x))\,d\xi\right\} < +\infty,$$

$$\sup_{[a,b]}\left\{|\alpha'(x)\beta'(\alpha(x))|\exp\left[\int_c^d L_1(x,\eta)\,d\eta + \int_a^b L_2(\xi,\alpha(x))\,d\xi\right]\right\} < 1, \tag{1}$$

e tali che, posto $\Delta_1 f=f(x,y,z,p_2,q)-f(x,y,z,p_1,q)$, $\Delta_2 f=f(x,y,z,p,q_2)-f(x,y,z,p,q_1)$, si abbia

$$\begin{aligned}
&\Delta_1 f \leqq L_1(x,y)(p_2-p_1) && \text{se } y>\alpha(x),\ p_2-p_1>0,\\
&\Delta_1 f \geqq -L_1(x,y)(p_2-p_1) && \text{se } y<\alpha(x),\ p_2-p_1>0,\\
&\Delta_2 f \leqq L_2(x,y)(q_2-q_1) && \text{se } x>\beta(y),\ q_2-q_1>0,\\
&\Delta_2 f \geqq -L_2(x,y)(q_2-q_1) && \text{se } x<\beta(y),\ q_2-q_1>0.
\end{aligned} \tag{2}$$

Successivamente, valendosi di questo risultato e di un procedimento di prolungamento dovuto a G. Stampacchia [Giorn. Mat. Battaglini (4) **3** (79) (1950), 66–85; MR **11**, 668] l'autore prova Teorema II che il problema ha almeno una soluzione in R se in luogo delle (1) vale la

$$|\beta(\alpha(x))| \leqq \theta|x| \tag{1'}$$

per qualche $\theta\in(0,1)$, e se invece delle (2) si suppone che per ogni costante $N>0$ esistano due funzioni $N_1(y)$ ed $N_2(x)$, con $N_1(y)\geqq 0$ sommabile e tale che

$$\Delta_1 f \leqq N_1(y)(p_2-p_1) \quad \text{per } y>\alpha(x),\ -N\leqq p_1<p_2\leqq N,$$

$$\Delta_1 f \geqq -N_1(y)(p_2-p_1) \quad \text{per } y<\alpha(x),\ -N\leqq p_1<p_2\leqq N,$$

ed $N_2(x)\geqq 0$ sommabile e tale che

$$\Delta_2 f \leqq N_2(x)(q_2-q_1) \quad \text{per } x>\beta(y),\ -N\leqq q_1<q_2\leqq N,$$

$$\Delta_2 f \geqq -N_2(x)(q_2-q_1) \quad \text{per } x<\beta(y),\ -N\leqq q_1<q_2\leqq N.$$

Con lo stesso metodo di prolungamento, si può provare infine (Teorema III) l'esistenza di soluzioni in R nelle ipotesi (1'), (2), ed altre riguardanti la posizione del punto 0 in R.

Il Teorema I estende un risultato di Z. Szmydt [Bull. Acad. Polon. Sci. Cl. III **5** (1957), 571–575; MR **19**, 748; 1ª recensione]; i teoremi II e III sono indipendenti da un recente risultato di J. Kisynski [Ann. Univ. Mariae Curie-Skłodowska. Sect. A **11** (1957), 73–112; MR **21** #5818].

R. Conti (Florence)

9725:

Guglielmino, Francesco. Sul problema di Darboux. Ricerche Mat. 8 (1959), 180–196.

Il risultato principale del lavoro è il seguente (Teorema II). Il problema di Darboux

$$z_{xy} = f(x, y, z, z_x, z_y), \quad z(x, y_0) = \sigma(x), \quad z(x_0, y) = \tau(y)$$

ammette nel rettangolo R: $x_0 \leq x \leq x_0+a$, $y_0 \leq y \leq y_0+b$ $(a, b>0)$, almeno una soluzione $z(x, y)$ continua con le derivate z_x, z_y, z_{xy}, se sono soddisfatte le seguenti condizioni: (i) $\sigma(x)$ e la derivata $\sigma'(x)$ siano continue in $x_0 \leq x \leq x_0+a$; $\tau(y)$ e la derivata $\tau'(y)$ siano continue in $y_0 \leq y \leq y_0+b$; sia $\sigma(x_0)=\tau(y_0)$; (ii) $f(x, y, z, p, q)$ sia continua nell'insieme S: $(x, y) \in R$, $|z|$, $|p|$, $|q| < +\infty$; sia $|f(x, y, z, p, q)| \leq K$ in S, con K costante; (iii) posto

$$K_1 = \max_R |\sigma(x)+\tau(y)-\sigma(x_0)| + Kab,$$

$$K_2 = \max_{[x_0, x_0+a]} |\sigma'(x)| + Kb,$$

$$K_3 = \max_{[y_0, y_0+b]} |\tau'(y)| + Ka,$$

per ogni coppia (x, y, z, p_1, q), (x, y, z, p_2, q) dell'insieme S^*: $(x, y) \in R$, $|z| \leq K_1$, $|p| \leq K_2$, $|q| \leq K_3$, con $p_1 < p_2$, si abbia

$$f(x, y, z, p_2, q) - f(x, y, z, p_1, q) \leq \psi^*(x, y, p_2-p_1),$$

essendo $\psi^*(x, y, u)$ una funzione non negativa definita per $(x, y) \in R$, $u \geq 0$, continua in u per ogni $(x, y) \in R$, misurabile in y per ogni $u \geq 0$ ed ogni $x \in [x_0, x_0+a]$. Esista inoltre una funzione $\varphi(x, y)$ definita in R, sommabile in $y \in [y_0, y_0+b]$, tale che $\psi^*(x, y, u) \leq \varphi(x, y)$ per $(x, y) \in R$, $0 \leq u \leq 2K_2$. Infine per ogni $\alpha \in (0, b]$, l'unica soluzione dell'equazione integrale $u(x, y) = \int_{y_0}^{y} \psi^*(x, \eta, u(x, \eta))d\eta$ nel rettangolo $x_0 \leq x \leq x_0+a$, $y_0 \leq y \leq y_0+\alpha$ sia la funzione ivi identicamente nulla; (iv) comunque si assegni $z(x, y)$ nella famiglia delle funzioni continue in R insieme con $z_x(x, y)$, e tali che

$$z(x, y_0) = \sigma(x), \quad z(x_0, y) = \tau(y),$$

$$|z_x(x, y_1) - z_x(x, y_2)| \leq K|y_1 - y_2|$$

per ogni coppia (x, y_1), $(x, y_2) \in R$, il problema iniziale

$$w_x = f(x, y, z(x, y), z_x(x, y), w), \quad w(x_0, y) = \tau'(y),$$

abbia in R una sola soluzione.

L'autore mostra che l'enunciato precedente include quelli di vari teoremi di esistenza precedentemente noti, di T. Satō [Mem. Fac. Sci. Kyūsyū Imp. Univ. A **2** (1941), 107–123; MR **3**, 122], di C. Ciliberto [Ricerche Mat. **4** (1955), 15–29; MR **17**, 621], di Z. Szmydt [Bull. Acad. Polon. Sci. Cl. III **4** (1956), 67–72; MR **19**, 39], e quindi anche di Ph. Hartman e A. Wintner [Amer. J. Math. **74** (1952), 834–864; MR **14**, 475].

Il risultato è ottenuto mediante l'applicazione di un noto teorema di J. Schauder sull'esistenza di punti fissi in una trasformazione continua di un opportuno spazio funzionale. *R. Conti* (Florence)

9726:

Bochner, S. Almost periodic solutions of the inhomogeneous wave equation. Proc. Nat. Acad. Sci. U.S.A. **46** (1960), 1233–1236.

La quasi-periodicità (q.p.) delle soluzioni dell'equazione omogenea delle onde, e le funzioni quasi periodiche astratte sono state considerate da Bochner per la prima volta, e già da molto tempo. Recentemente, il recensore ha dimostrato la quasi-periodicità delle soluzioni relativamente compatte dell'equazione delle onde non omogenea, con termine noto quasi periodico. Poi Amerio ha ottenuto la quasi-periodicità delle soluzioni soltanto limitate, e anche la quasi-periodicità degli integrali limitati di una funzione quasi periodica a valori in uno spazio di Hilbert.

L'oggetto del presente lavoro è di enunciare un teorema astratto chi contiene i due risultati di Amerio. Questo teorema si riferisce alla quasi-periodicità delle soluzioni deboli limitate di una equazione astratta $u'(t)=Cu(t)+f(t)$, $f(t)$ q.p., dove C è un operatore non limitato, soddisfacente a varie condizioni, fra le quali ce n'è una che assicura la conservazione dell'energia quando $f=0$. *S. Zaidman* (Milan)

9727:

Wells, C. P. Separability conditions for some self-adjoint partial differential equations. Amer. Math. Monthly **66** (1959), 684–689.

This interesting paper gives conditions under which second-order elliptic differential equations, in only two independent variables, but with variable coefficients, can be solved by separation of variables. The literature of separation distinguishes two definitions. One can require that the equation actually separates into two parts, so that the transformed operator $TL_{xy}=L_{uv}$ has the form $F(u, v)(L_u+L_v)$ (subscripts denoting the independent variables involved in the operator L). Or, one can require merely the existence of a complete family of separated solutions $U_a(u)V_a(v)$, where a is a parameter. The paper under review uses the first definition, which is somewhat more restrictive than the second.

Within this framework Theorem 1 gives necessary and sufficient conditions for separation of $L[\psi]=0$, where L is a self-adjoint operator of the kind described in the preceding paragraph. Theorem 2 gives the additional restrictions relevant to $L[\psi]+k\psi=0$. The author exhibits the co-ordinates in full detail for $(A\psi_x)_x+(A\psi_y)_y+k\psi=0$. The transformation is now given by the real and imaginary parts of $\psi(w)=\int^w x(\zeta)\exp(-\zeta^2)d\zeta$ where x is one of five specific functions.

R. M. Redheffer (Los Angeles, Calif.)

9728:

Montaldo, Oscar. Su un problema di valori al contorno nella teoria diffusiva dei reattori nucleari. Rend. Sem. Fac. Sci. Univ. Cagliari **28** (1958), 118–120.

Consider the parabolic system (1) $D_i\partial^2 u_i/\partial x^2 - h_i\partial u_i/\partial t + (-1)^i\varepsilon_i u_1 - (-1)^i\lambda_i u_2 = 0$ $(i=1, 2)$ where $\lambda_1 = k\varepsilon_2$, $\lambda_2 = \varepsilon_1$ and the D_i, ε_i, h_i and k are positive constants. These equations are considered in the rectangle $0<x<1$, $0<t<h$, together with the boundary conditions (2) $\partial u_i(0, t)/\partial x = 0$, $2D_i\partial u_i(1, t)/\partial x + u_i(1, t) = 0$ on $0 \leq t \leq h$, (3) $u_i(x, 0) = f_i(x)$ on $0 \leq x \leq 1$. The author proves uniqueness for (1)–(3), in case $0<k\leq 1$. He also reduces the system (1)–(3) to that of one parabolic equation of the fourth order. Existence then follows by results of B. Pini [Rend. Sem. Mat. Univ. Padova **27** (1957), 319–349, 387–410; MR **20** #7150]. *A. Friedman* (Minneapolis, Minn.)

9729:

Pogorzelski, W. Sur certaines propriétés des intégrales analogues aux potentiels et un problème aux limites pour l'équation parabolique. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **8** (1960), 53–58. (Russian summary, unbound insert)

In an earlier work [Ann. Polon. Math. **4** (1957), 61–92, 110–126; MR **21** #3656a, b], the author proved the existence of solutions $u(x, t)$ for a general class of mixed initial-boundary-value problems for the normal parabolic equation

$$\phi(u) \equiv \sum_{\alpha,\beta=1}^{n} a_{\alpha\beta}(X,t)\frac{\partial^2 u}{\partial x_\alpha \partial x_\beta} + \sum_{\alpha=1}^{n} b_\alpha(X,t)\frac{\partial u}{\partial x_\alpha} + c(X,t)u - \frac{\partial u}{\partial t} = F\left(X, t, u, \frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}\right)$$

with Hölder-continuous a, b, c, F and continuous initial and boundary values. Now he sketches an existence proof for an even larger set of initial and boundary conditions: the initial function is only required to be integrable, while the boundary values are allowed to become unbounded to the order $t^{-r'}$ $(r' < 1)$ as t approaches zero.

H. C. Kranzer (Garden City, N.Y.)

9730:

Hawlitschek, Kurt. Greensche Funktionen für Randwertaufgaben partieller Differentialgleichungen vom parabolischen Typ. Math. Ann. **140** (1960), 65–75.

The author considers boundary-value problems for a class of parabolic equations

$$u_t = A(x,t)u_{xx} + B(x,t)u_x \tag{1}$$

in a region R, under the assumption that there is a transformation

$$X = v(x,t) \qquad T = w(t), \tag{2}$$

with v a solution of (1) in R, which transforms (1) into the heat equation and R into a trapezoidal region

$$m_1 T + n_1 < X < m_2 T + n_2, \quad t_1 < T < t_2. \tag{3}$$

A Green's function Γ for the original boundary-value problem is expressed in terms of a Green's function Γ_0 for the transformed problem by the relation

$$\Gamma(x,t,y,s) = v_y(y,s)\Gamma_0(X,T,Y,S). \tag{4}$$

Since Γ_0 is given in terms of Jacobi theta series, this leads to an explicit formula for Γ. Examples are given at the end of the paper.

J. Elliott (New York)

9731:

Piskorek, A. Propriétés d'une intégrale de l'équation parabolique dans un domaine non cylindrique. Ann. Polon. Math. **8** (1960), 125–137.

Let D be a cylindrical domain in a Euclidean $(X, t) = (x_1, \cdots, x_n, t)$-space, i.e., D is an open set in the half-space $t > 0$ with a boundary consisting of a bounded n-dimensional X-set in the plane $t = 0$ and a lateral boundary surface s having a continuous (inward) normal vector N nowhere parallel to the t-axis. Let $D(T)$ be the portion of D between the planes $t = 0$ and $t = T > 0$ and $s(T)$ the lateral boundary of $D(T)$. It is supposed that $N = N(Q, \tau)$ is uniformly Hölder-continuous with respect to $(Q, \tau) \in s(T)$. Consider a parabolic equation

$$\sum_{i,j=1}^{n} a_{ij}(X,t)\partial^2 u/\partial x_i \partial x_j + \sum_{k=1}^{n} b_k(X,t)\partial u/\partial x_k + c(X,t)u - \partial u/\partial t = 0 \tag{1}$$

with coefficients continuous in a domain containing $D(T)$ in its interior. In addition, it is supposed that the a_{ij} are uniformly Hölder-continuous with respect to (X, t) and b_k, c with respect to X. The paper deals with the solution $u = U(X, t)$ of (1) in $D(T)$ given by

$$U(X,t) = \int_{s(t)} \Gamma(X,t;Q,\tau)\frac{\varphi(Q,\tau)}{\sin(N,t)}\,ds,$$

where $(X, t) \in D(T)$, where $(Q, \tau) \in s(t)$, $N = N(Q, \tau)$, ds is the element of surface on s at (Q, τ), $\varphi(Q, \tau)$ is a given bounded, measurable function on $s(T)$ and Γ is a fundamental solution of (1). The author establishes the uniform Hölder-continuity of U and, when φ is continuous, the existence and properties of a certain "transversal" derivative of U at points of $s(T)$. His results are extensions of those of Pogorzelski [Ricerche Mat. **5** (1956), 25–57; Ann. Polon. Math. **4** (1957), 61–92; MR **18**, 47; **21** #3656a] for the case when $D(T)$ is a product set $\Omega \times (0, T)$.

P. Hartman (Baltimore, Md.)

9732:

Kaminin, L. I. La solution d'après la méthode Fourier du premier problème extrêmal pour l'équation de la conductibilité de la chaleur. Bul. Univ. Shtetëror Tiranë. Ser. Shk. Nat. **1959**, no. 1, 16–33. (Albanian. French summary)

The equation $u_t = u_{xx} + \lambda f(x, t, u)$, $u(0, t) = u(\pi, t) = 0$, $u(x, 0) = g(x)$, where f has continuous second derivatives, is solved for small λ by a method involving the Schauder fixed-point theorem as well as Fourier expansion.

P. Ungar (New York)

9733:

Čifligu, A.; Maslennikova, V. N.; Kamynin, L. I. On the applicability of Fourier's method to the solution of the first boundary value problem for a quasilinear parabolic equation. Dokl. Akad. Nauk SSSR **130** (1960), 738–741 (Russian); translated as Soviet Math. Dokl. **1**, 98–101.

Brief summary and extension of the previous work [#9732] to the equation with u_{xx} replaced by $(p(x)u_x)_x - q(x)u$.

P. Ungar (New York)

9734:

Nickel, Karl. Ein Eindeutigkeitssatz für instationäre Grenzschichten. Math. Z. **74** (1960), 209–220.

The author derives (by a comparison argument) a uniqueness theorem for solutions of quasi-linear parabolic equations satisfying a nonlinear Neumann's condition on the lateral boundary of the (non-cylindrical, in general) domain. He then applies his theorem to prove the uniqueness of solutions for the Prandtl equations $\partial u/\partial t + u\partial u/\partial x + v\partial u/\partial y = U_0$, $\partial u/\partial x + \partial v/\partial y = 0$ (U_0 is given) with appropriate initial and boundary conditions. One of the assumptions made, in the uniqueness theorem, is that $u \geqq 0$.

A. Friedman (Minneapolis, Minn.)

9735:

Uchiyama, Moritsune. A method of solution of linear partial differential equations of the second order with function coefficients. Proc. Japan Acad. **35** (1959), 316–320.

The paper proposes a method for introducing a change of coordinates $x \to \xi$ which, under certain conditions, reduces a partial differential equation of the form

$$g_{ij}(x)\partial^2 u/\partial x^i \partial x^j + b^j(x)\partial u/\partial x^j + c(x)u = f(x)$$

into a normal form $\varepsilon_i \partial^2 u/\partial \xi^i \partial \xi^i + \bar{b}^j(\xi)\partial u/\partial \xi^j + \bar{c}(\xi)u = \bar{f}(x)$ with $\varepsilon_i = \pm 1$. {Neither the conditions on the coefficients nor the meaning given to "change of coordinates" is clear to the reviewer.} *P. Hartman* (Baltimore, Md.)

9736:

Suyama, Yukio; Ono, Akira. On the local regularity of generalized solutions of linear elliptic partial differential equations. Mem. Fac. Sci. Kyushu Univ. Ser. A **14** (1960), 134–138.

Let L be an elliptic partial differential operator of order $2m$ in a domain G in E^n with coefficients in $C^m(G)$. The authors prove the following. I. If $f \in L^p(G)$, then in any compact subdomain G_1 of G the $(2m-1)$st derivatives of the solution u of $Lu=f$ satisfy an $L^q(G_1)$ "Hölder" condition with exponent s provided that $1/q > 1/p - (1-s)/n$. II. If f also satisfies an $L^p(G')$ "Hölder" condition $(G_1 \subset G' \subset G)$ with exponent α, then the same is true of the $2m$th derivatives of u.

M. Schechter (New York)

9737:

Huet, Denise. Phénomènes de perturbation singulière dans les problèmes aux limites. Ann. Inst. Fourier. Grenoble **10** (1960), 61–150.

Let B_ε be a family of partial differential operators such that a certain boundary problem is well posed for each equation (1) $B_\varepsilon u = f$. If the B_ε converge to an operator B of lower order as $\varepsilon \to 0$, do the solutions u_ε of (1) converge in any sense to the solution of $Bu=f$? This question is studied in the present paper for boundary problems which are elliptic in the sense of Lions [Acta Math. **94** (1955), 13–153; MR **17**, 745]. Under suitable hypotheses some convergence theorems are established. Several examples are given. The results are then applied to mixed problems in the sense of Lions [loc. cit.].

M. Schechter (New York)

9738:

Il'in, V. P. Some inequalities in function spaces and their application to the investigation of the convergence of variational processes. Trudy Mat. Inst. Steklov. **53** (1959), 64–127. (Russian)

The first section of this paper is devoted to problems of the following type: given a function $f(P)$ in an n-dimensional domain D with a reasonably well-behaved frontier Γ, and given estimates for the L^p norm $\|f\|_p$ of f and for the L^p norms of its lth-order derivatives, to obtain estimates for the values of f and its kth-order derivatives for $k \leq l-1$, or for their norms of order higher than $\max(p, s)$, or for more elaborate norms involving Hölder conditions. In § 2 these results are applied to obtain results on the convergence of a sequence $u_i(P)$ to a function $u(P)$, either uniformly in D or with respect to some norm, given estimates for $\|u-u_i\|_p$ and for norms involving the lth-order derivatives of $u-u_i$. Results on the convergence of the sequences of derivatives are also obtained.

In § 3 the author deduces conditions under which one can obtain uniform convergence (or some other strong form of convergence) of a minimizing sequence of functions when the direct methods of the calculus of variations are applied to boundary-value problems for second-order elliptic partial differential equations and for the biharmonic equation. In § 4 special results are obtained for the cases when polynomials or other elementary functions are used to form the minimizing sequence (as, for instance, in some forms of the Ritz method). The paper ends with some results on the order of polynomial approximation to functions of certain special classes, and some applications of these results.

The results obtained are much too complicated to describe in detail here; one sample (part of theorem 1 of § 2) must suffice. Let (u_i) be a sequence of functions defined in D, and suppose that $\|u_i\|_p \leq A_i$, $\|[\sum |D_{x_j}{}^l|^2]^{1/2}\|_p \leq B_i$, the sum being taken over all the lth-order partial derivatives of u_i; suppose also that $0 \leq k \leq l-1$, $(l-k)p > n$, $A_i \to 0$ $(i \to \infty)$, and $A_i{}^{l-k-(n/p)} B_i{}^{k+(n/p)} \to 0$ $(i \to \infty)$; then every kth-order partial derivative of u_i is uniformly convergent to 0 in $D \cup \Gamma$.

Examples are given to indicate the extent to which the results are known to be best possible.

F. Smithies (Cambridge, England)

9739:

Friedman, Avner. On the regularity of the solutions of nonlinear elliptic and parabolic systems of partial differential equations. J. Math. Mech. **7** (1958), 43–59.

The author considers solutions to the nonlinear elliptic system of partial differential equations

$$(*) \qquad \Phi_j(x, u, \nabla u, \cdots) = 0$$

(here $x=(x_1, \cdots, x_r)$, $u=(u_1, \cdots, u_N)$, and "$\nabla u, \cdots$" denotes derivatives up to certain orders). "Elliptic" here means that the osculating linear system is elliptic in the sense of Douglis and Nirenberg [Comm. Pure Appl. Math. **8** (1955), 503–538; MR **17**, 743]. The Φ's as well as the u's are allowed to be complex.

The main results of the paper are regularity theorems for the system (*) which include, as special cases, the analyticity theorems proved by Petrovskiĭ and also Morrey, to the effect that if the Φ_j are analytic, then so are the u_j. The paper has novelty both in methods and results. First, whereas both Petrovskiĭ and Morrey prove analyticity by continuation into the complex plane, the present paper avoids this. Secondly, the results are stronger in the following sense: Certain regularity classes of (C^∞) functions are introduced (related to the quasi-analytic classes of Mandelbrojt). Now, if the Φ's belong to one such class, then if the u's possess a certain number of derivatives, they belong to another such class. In particular it follows that solutions to (*) possess the unique continuation property under conditions somewhat weaker than the analyticity of the Φ_j.

In the proof the author uses estimates for linear elliptic systems previously derived by himself [Proc. Amer. Math. Soc. **8** (1957), 418–427; MR **20** #3376] and Morrey and Nirenberg [Comm. Pure Appl. Math. **10** (1957), 271–290; MR **19**, 654]. Under the assumption that

the system (*) is strongly elliptic, an analogous regularity theorem is proved "up to the boundary". It includes as a special case the result previously proved by Morrey to the effect that if the system (*) is strongly elliptic and the φ_j, the boundary and Dirichlet data are all analytic, then the solutions u can be continued analytically across the boundary.

Some results are stated for parabolic systems, but their proofs are omitted. *W. Littman* (Minneapolis, Minn.)

9740:

Kasahara, Koji; Yamaguti, Masaya. Strongly hyperbolic systems of linear partial differential equations with constant coefficients. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **33** (1960/61), 1–23.

The main result of this paper was announced in Proc. Japan Acad. **35** (1959), 547–550 [MR **22** #3881]. *R. W. McKelvey* (Los Angeles, Calif.)

9741:

Zaidel', R. M.; Semendyaev, K. A. Limiting solutions of a certain parabolic equation. Dokl. Akad. Nauk SSSR **133** (1960), 311–314 (Russian); translated as Soviet Physics. Dokl. **5** (1961), 705–708.

A method is described for obtaining particular solutions of $a^{-1}\partial F/\partial t = F\Delta Fk^{-1}(\nabla F)^2$ having certain properties. *M. Schechter* (New York)

9742:

Milicer-Grużewska, H. Un théorème limite sur la dérivée de l'intégrale de Poisson-Weierstrass généralisée. Ann. Polon. Math. **7** (1959), 71–80.

This paper contains the proof of a theorem previously announced in Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **6** (1958), 131–133 [MR **20** #5342]. *J. Elliott* (New York)

9743:

Kamynin, L. I.; Maslennikova, V. N. The solution of the first boundary problem in the large for a quasilinear parabolic equation. Dokl. Akad. Nauk SSSR **137** (1961), 1049–1052 (Russian); translated as Soviet Math. Dokl. **2**, 405–408.

The first boundary-value problem for the equation in question,

$$(*)\qquad \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x,t,u)\frac{\partial u}{\partial x_i} - \frac{\partial u}{\partial t} = f(x,t,u,\nabla u),$$

$$\nabla u = (\partial u/\partial x_1, \partial u/\partial x_2, \cdots, \partial u/\partial x_n),$$

was proved by A. Friedman [J. Math. Mech. **9** (1960), 539–556; MR **22** #6929] to have a local solution in the linear case (i.e., when $b_i(x,t,u) \equiv b_i(x,t)$) by means of a $(1+\delta)$-estimate based on the fundamental solution for parabolic equations. In the present paper theorems on the existence and uniqueness of the solution in the large when (*) is quasilinear are given. The proofs are not included, but the existence proof is asserted to rest on Friedman's estimates and the Schauder fixed-point theorem. *R. N. Goss* (San Diego, Calif.)

9744:

Agaev, G. N.; Mamazov, G. K. Solution of a mixed problem for a non-linear parabolic equation. Dokl. Akad. Nauk Azerbaĭdžan. SSR **14** (1958), 505–510. (Russian. Azerbaijani summary)

The author is concerned with a nonlinear parabolic equation of the type

$$(1)\qquad \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = f(t,x,u)$$

in a region $D\,[0<t<T,\ \varphi_1(t)<x<\varphi_2(t)]$ with the boundary condition

$$(2)\qquad u|_\Gamma = 0.$$

We denote $D_1 = D\times(-L, L)$, where L is a positive constant. Let $f(t,x,u)$ be continuous in the closed region $\bar{D}_1$ and $|f(t,x,u)-f(t,x,v)| \le A|u-v|/t^\alpha$ $(\frac{1}{2} \le \alpha < 1)$, $|f(t,x,0)| \le A_1/t^\alpha$ (A, A_1 positive constants); further let $\varphi_1(t)$, $\varphi_2(t)$ be continuous functions for $0 \le t \le T$ and

$$|\varphi_2(t)-\varphi_1(t)| < Kt^{\alpha-1/2},$$

where $K < \min(2/A\sqrt{\pi},\ 2L/A_2\sqrt{\pi})$, $A_2 = AL+A_1$. Then there exists one and only one (classical) solution of the problem (1), (2).

The theorem is proved with help of the Green's function; the problem (1), (2) is transformed into an equivalent one of an integral equation, which is solved by the method of successive approximations.

Finally, remarks are presented about the validity of the result for the case where the right side of equation (1) has the form $f(t,x,u,\partial u/\partial x)$, and the case of equations containing functions with values in a Banach space. *K. Rektorys* (Prague)

9745:

Montaldo, Oscar. Sulle equazioni paraboliche lineari d'ordine $2n$ in due variabili. Rend. Sem. Fac. Sci. Univ. Cagliari **28** (1958), 111–117.

Consider the differential equation

$$(1)\qquad \sum_{k=0}^{n} (-1)^k a_k \partial^{2n-k}u(x,y)/\partial x^{2n-2k}\partial y^k = 0,$$

where a_k are positive constants such that all the roots of the polynomial $\sum (-1)^k a_k \lambda^{n-k}$ have positive real parts. Let $x = \chi_i(y)$ $(0 \le y \le H,\ i = 1, 2)$ be smooth curves and $\chi_1 < \chi_2$. Consider the boundary conditions

$$(2)\qquad \sum \alpha_{hk}^{(i)}(y)\partial^{h+k}u/\partial x^{h+k} = f_h^{(i)}(y)$$

on $x = \chi_i(y)$ $(i = 1, 2;\ h = 0, 1, \cdots, n-1)$,

$$(3)\qquad \alpha_h(x)\partial^{h-1}u/\partial y^{h-1} + \beta_h(x)\partial^h u/\partial y^h = \varphi_h(x)$$

on $\chi_1(0) \le x \le \chi_2(0)$, $y = 0$ $(h = 1, 2, \cdots, n)$. Generalizing his previous results [same Rend. **26** (1956), 1–6; **28** (1958), 1–8; MR **18**, 314; **21** #4299] the author proves, under some smoothness assumptions on the coefficients of (2), (3), that the problem (1), (2), (3) has a solution. The method is by reducing the problem to that of solving Volterra-type integral equations. *A. Friedman* (Minneapolis, Minn.)

9746:

Mihaĭlov, V. P. A mixed problem for a parabolic system on a plane. Dokl. Akad. Nauk SSSR **126** (1959), 1199–1202. (Russian)

The author presents the main results of his work, which will be published later. He investigates a parabolic system

$$\left(\frac{\partial}{\partial t}-A\left(x,t,\frac{\partial}{\partial x}\right)\right)u = f(x,t),$$

where $u(x,t)=(u_1(x,t),\cdots,u_N(x,t))$, $f(x,t)=(f_1(x,t),\cdots,f_N(x,t))$, and

$$A\left(x,t,\frac{\partial}{\partial x}\right) = \sum_{k=0}^{2p} A_k(x,t)\frac{\partial^{2p-k}}{\partial x^{2p-k}}.$$

The $A_k(x,t)$ are square matrices of the Nth order with sufficiently smooth elements. The initial and boundary conditions are: $u|_{t=0}=\varphi(x)$, $\partial^i u/\partial x^i|_{x=0}=\chi_i(t)$, $\partial^i u/\partial x^i|_{x=1}=\psi_i(t)$, $i=0,\cdots,p-1$. Lemmas and theorems are presented concerning existence and unicity of the solution and its properties (continuous dependence on initial and boundary conditions, etc.). The work is based on properties of the corresponding Green's matrix, which are discussed in detail. *K. Rektorys* (Prague)

9747:

Kim, E. I.; Ivanova, L. P. The mixed boundary value problem for a certain system of parabolic differential equations. Dokl. Akad. Nauk SSSR **126** (1959), 1183–1186. (Russian)

A system of equations

$$(1)\quad \partial u_i/\partial t = \sum a_{ik}\Delta u_k \qquad (i=1,2,\cdots,n;\ \Delta=\partial^2/\partial x^2+\partial^2/\partial y^2),$$

where the a_{ik} are constant complex coefficients, is investigated, supposing that all roots of the equation $|A-\lambda E|=0$ ($A=\|a_{ij}\|$, E is the unit matrix) are different and $\operatorname{Re}\lambda>0$. A solution of the system (1) in a region D with sectionally differentiable boundary C is to be found, with initial conditions

$$(2)\quad u_i(x,y,0)=f_i(x,y)$$

and boundary conditions

$$(3)\quad u_i|_C=\varphi_i(s,t)$$

or

$$(4)\quad \left(\frac{\partial u_i}{\partial v}+\sum_{k=1}^{n} a_{ik}(s,t)u_k\right)\Big|_C=\psi_i(s,t) \qquad (i=1,2,\cdots,n).$$

The author first constructs fundamental solutions for the system (1) and transforms the problem (in a way analogous to that using temperature potentials) into the problem of solving a system of integral equations. Under some suppositions about the functions f_i, φ_i or ψ_i, these equations can be solved by the method of successive approximations. *K. Rektorys* (Prague)

9748:

Saaty, Thomas L. An application of semi group theory to the solution of a mixed type partial differential equation. Univ. e Politec. Torino. Rend. Sem. Mat. **18** (1958/59), 53–76.

This paper treats the Tricomi equation $xU_{xx}+U_{tt}=0$ in the region $x<0$, $t>0$, and obtains the solution to the Cauchy problem in which $U(x,0)$, $U_t(x,0)$ are prescribed functions in $C[-\infty,0]$. This is done by changing to a system of two equations in the unknowns U and U_t, and then describing the solution to be a semi-group of operators with the infinitesimal generator that carries the pair (f_1,f_2) of functions in $C[-\infty,0]$ into the pair $(-xf_2'',f_1)$. It is shown that all pure imaginary numbers belong to the continuous spectrum and that all other complex numbers belong to the resolvent set. The resolvent is explicitly represented, and the complex inversion formula is used to find a representation for the desired semi-group. This representation is written out explicitly for those values of x and t satisfying $0<t<2(-x)^{1/2}$ and becomes an integral operator whose kernel involves the Legendre functions of order 1/2. The paper concludes by noting that an immediate generalization is possible to the equation $x^{2\nu+1}F_{xx}+F_{tt}=0$, $\nu=0,1,\cdots$.

The author begins ab initio, and the paper is perhaps a good introduction to the semi-group technique of treating partial differential equations.

G. Hufford (Seattle, Wash.)

POTENTIAL THEORY

See also 9596.

9749:

Brelot, M. ★Lectures on potential theory. Notes by K. N. Gowrisankaran and M. K. Venkatesha Murthy. Lectures on Mathematics, 19. Tata Institute of Fundamental Research, Bombay, 1960. ii+155 pp.

Cet ouvrage complète le livre récent de M. Brelot [*Eléments de la théorie classique du potentiel*, Centre de Documentation Universitaire, Paris, 1959; MR **21** #5099] sur deux sujets importants: les théorèmes de convergence pour les noyaux réguliers, et la nouvelle axiomatique des fonctions harmoniques et surharmoniques élaborée ces toutes dernières années par l'auteur. Il est divisé en quatre parties, de dimensions très inégales. Une très courte partie I est consacrée au "lemme topologique" de Choquet concernant les enveloppes inférieures de certaines familles de fonctions numériques. La partie II est un exposé de la théorie de la capacité suivant Choquet; on y donne des démonstrations complètes des théorèmes de capacitabilité sous diverses hypothèses simples se présentant naturellement en théorie du potentiel. La partie III, consacrée aux noyaux réguliers et aux extensions à ces noyaux des théorèmes de convergence de Brelot-Cartan, reprend et développe le chapitre VI de l'ouvrage cité de M. Brelot, avec des précisions nouvelles sur les ensembles exceptionnels. Enfin dans la partie IV, qui est plus longue à elle seule que les trois autres réunies, l'auteur donne un exposé complet de sa nouvelle axiomatique des fonctions harmoniques et surharmoniques sur un espace localement compact connexe, développant des résultats annoncés dans diverses notes récentes [M. Brelot, C. R. Acad. Sci. Paris **245** (1957), 1688–1690; **246** (1958), 2334–2337, 2709–2712; MR **21** #5090a, b, #5095; R. M. Hervé et M. Brelot, ibid. **247** (1958), 1956–1959; MR **21** #5096; R. M. Hervé, ibid. **248** (1959), 179–181; MR **21** #5097; à signaler aussi deux exposés, moins élaborés, par M. Brelot, *Séminaire de théorie du potentiel*, 1re *année*: *1957*, Paris, 1958; MR **21** #5093; et 2*me année*: *1958*]. Cet exposé comprend l'étude

des familles de fonctions "harmoniques" et "surharmoniques", des "potentiels" (fonctions surharmoniques positives dont la plus grande minorante harmonique est nulle; il s'agit donc d'une théorie du potentiel "sans noyau"), du problème de Dirichlet, des ensembles polaires, des théorèmes de convergence, de l'extrémisation, de l'effilement, de la frontière de Martin et des représentations intégrales des fonctions surharmoniques positives. Les interprétations probabilistes sont délibérément laissées de côté. Les axiomes choisis sont discutés en détail; ils sont très naturels et semblent revêtir une forme à peu près définitive; on souhaiterait cependant que des exemples plus nombreux précisent la portée exacte de cette très élégante théorie; les recherches les plus récentes de Mme Hervé, mentionnées dans le dernier chapitre, apporteront peut-être une réponse à cette question, dans laquelle il reste encore bien des problèmes ouverts.

Bien que cet ouvrage soit consacré à des sujets d'actualité et assez techniques, sa lecture ne suppose aucune connaissance préalable de théorie du potentiel, mais seulement quelques éléments de topologie générale et de théorie de la mesure. Il permettra au lecteur non spécialisé de s'initier facilement à quelques-uns des aspects les plus récents d'une théorie attrayante, dont l'importance s'affirme chaque jour davantage, mais à laquelle n'était consacré aucun traité didactique moderne.

J. Deny (Paris)

9750:

Fufaev, V. V. On the Dirichlet problem for regions with corners. Dokl. Akad. Nauk SSSR **131** (1960), 37–39 (Russian); translated as Soviet Math. Dokl. **1**, 199–201.

Let Ω be a plane region bounded by a sufficiently smooth curve Γ, let $f(s)$ be a function defined on Γ, where s denotes arc-length measured along Γ, and let $U(x, y)$ be a function which is harmonic in Ω and equal to $f(s)$ on Γ. A result, relating differentiability or smoothness properties of $U(x, y)$ in Ω to corresponding properties of $f(s)$ on Γ, has been obtained by N. P. Mozžerova [same Dokl. **118** (1958), 636–638; MR **20** #5370], S. M. Nikol'skiĭ [Mat. Sb. (N.S.) **43 (85)** (1957), 127–144; MR **20** #5368a], and Ya. S. Bugrov [Dokl. Akad. Nauk SSSR **115** (1957), 639–642; MR **20** #183].

When Γ is piecewise smooth, the above result does not hold. The author continues this study and gives, without proof, relations between the smoothness of $U(x, y)$ and $f(s)$ when Γ has a corner of angle π/j and satisfies certain smoothness condition except at this corner. When j is a positive integer, these relations are sharp and extend results of Nikol'skiĭ [op. cit.] for the case $j=2$.

F. W. Gehring (Ann Arbor, Mich.)

9751:

Martensen, Erich; von Sengbusch, Kurt. Über die Randkomponenten ebener harmonischer Vektorfelder. Arch. Rational Mech. Anal. **5**, 46–75 (1960).

Die Verf. diskutieren die folgende für Strömungsvorgänge wichtige Aufgabe: Es ist ein überall ausserhalb einer glatten Jordankurve C harmonisches, d.i. wirbel- und quellenfreies Vektorfeld aus seinen Normal- bzw. Tangentialkomponenten längs C zu bestimmen. Besser als im Reellen, wie die Verf. es tun, lässt sich die Situation im Komplexen beschreiben. Sind a und b die x- bzw. y-Komponenten des Vektors, so ist der "komplexe" Vektor $f=a-ib$ eine ausserhalb C holomorphe und auf C noch stetige Funktion des Ortes $z=x+iy$, und somit

$$f(z)-f(\infty) = \frac{1}{2\pi i}\oint_C (\zeta-z)^{-1}f(\zeta)\,d\zeta.$$

Beachtet man die Gleichung $\partial \log|1/(\zeta-z)|/\partial z = \frac{1}{2}(\zeta-z)^{-1}$ so folgt $f(z)=f(\infty)+\partial\phi(z)/\partial z$ mit

$$\phi(z) = (\pi i)^{-1}\oint f(\zeta)\,d\zeta\cdot\log|(\zeta-z)^{-1}| = (\pi i)^{-1}\oint \log|(\zeta-z)^{-1}|(v_t+iv_n)\,ds,$$

wenn v_t und v_n die Tangential- bzw. Normalkomponenten des Feldes auf C bezeichnen. Formuliert man, mit Hilfe der Sprungrelation für das logarithmische Potential einer einfachen Schicht, dass f auf C noch stetig ist, so ergibt sich zur Bestimmung des Feldes längs C aus v_n bzw. v_t je eine Integralgleichung. Diese wird diskutiert und der numerischen Behandlung (diskrete Approximation) angepasst. Die Durchrechnung eines "verallgemeinerten Joukowski-Profils" ergibt sehr gute Uebereinstimmung mit der exakten Lösung. Die Methode eignet sich auch zur Behandlung von Gitterprofilen. *A. Pfluger* (Zürich)

9752:

Kishi, Masanori. A continuity theorem in the potential theory. Proc. Japan Acad. **36** (1960), 65–67.

Résultats concernant l'effilement et la topologie fine dans la théorie du potentiel par rapport à un noyau "régulier" symétrique sur un espace localement compact séparable. En particulier, énoncé d'un théorème de "quasi continuité locale" pour les potentiels: tous les points x de l'espace, sauf ceux d'un ensemble exceptionnel qui est polaire, possèdent la propriété suivante: il existe un ouvert $G(x)$ effilé en x tel que la restriction du potentiel considéré au complémentaire de $G(x)$ soit continue au point x.

J. Deny (Paris)

FINITE DIFFERENCES AND FUNCTIONAL EQUATIONS

See also 9467.

9753:

Levy, H.; Lessman, F. ★Finite difference equations. The Macmillan Company, New York, 1961. vii+278 pp. $5.50.

A reprint, without apparent change, of a book previously published in England [I. Pitman, London, 1958] and reviewed in MR **21** #5827. *T. Fort* (Coral Gables, Fla.)

9754:

Koval', P. I. Asymptotic behaviour of the solutions of almost triangular systems of differential and difference linear equations. Dokl. Akad. Nauk SSSR **124** (1959), 1203–1206. (Russian)

Consider two systems of difference equations of the form (1) $Y_{s+1}=R_sY_s$ and (2) $X_{s+1}=(R_s+C_s)X_s$; Y_s, X_s are unknown matrices, R_s and C_s are given, and R_s is triangular ($s=s_0, s_0+1, \cdots$). Some sufficient conditions for R_s and C_s are given in order that (2) have a solution

of the form $X_s = Y_s + H_s F_s$, where Y_s is a solution of (1), F_s is diagonal and composed of diagonal elements of Y_s, and $H_s \to 0$ as $s \to \infty$. A similar result is presented for systems of differential equations. Conditions are also given for transforming an arbitrary system of difference (or of differential) equations into a triangular one. Theorems are illustrated by several examples. The paper does not contain any proofs. *C. Olech* (Baltimore, Md.)

9755:

Kurepa, S. On some functional equations in Banach spaces. Studia Math. **19** (1960), 149–158.

This is a further study of the functional equation $F(x+y)+F(x-y)=2F(x)F(y)$, which the author previously examined on the hypothesis that F is a mapping of the real line into the class of continuous linear operators in a Hilbert space [Canad. J. Math. **12** (1960), 45–50; MR **22** #152]. In this paper a miscellany of results is given, dealing with the aforementioned functional equation and others related to it, all the equations being in some way traceable back to the sine and cosine functions. In some cases F is taken to be a complex-valued functional defined on a Banach space.

A. E. Taylor (Los Angeles, Calif.)

9756:

Chu, Pei-Chang. A system of functional equations. Acta Math. Sinica **6** (1956), 263–269. (Chinese. English summary)

Let x and ξ lie in $I=(0, \infty)$. Write

$$B_k^l = \frac{(l+k)!}{(l-k)!k!2^k},$$

so that $y_l(x)=\sum_{k=0}^{l} B_k^l x^k$ is the lth Bessel polynomial. The author studies the system of $5n+5$ functional relations

$$F_{l+1/2}(x)F_{-l-1/2}(\xi)+F_{l+1/2}(\xi)F_{-l-1/2}(x) = \left(\frac{2}{\pi}\right)^{1/2} \sum_{k=0}^{l} (-1)^k B_k^l \left(\frac{x+\xi}{x\xi}\right)^{k+1/2} F_{k+1/2}(x+\xi),$$

$$F_{-l-1/2}(x)F_{-l-1/2}(\xi)-F_{l+1/2}(x)F_{l+1/2}(\xi) = (-1)^l \left(\frac{2}{\pi}\right)^{1/2} \sum_{k=0}^{l} B_k^l \left(\frac{x+\xi}{x\xi}\right)^{k+1/2} F_{-k-1/2}(x+\xi),$$

$$F_{l+1/2}^2(x)+F_{-l-1/2}^2(x) = \frac{2}{\pi} \sum_{k=0}^{l} B_k^l B_k^k x^{-2(l+1/2)},$$

$$\lim_{x\to 0} x^{-(l+1/2)} F_{l+1/2}(x) = \left(\frac{2}{\pi}\right)^{1/2} ((2l+1)B_l^l)^{-1},$$

$$x^{l+1/2} F_{-l-1/2}(x) = O(1),$$

where $l=0, 1, \cdots, n$, and he shows that its only solution is given by the system of Bessel functions $\{J_{\mp(l+1/2)}(x)\}$. The proof uses induction for n. *K. Mahler* (Manchester)

SEQUENCES, SERIES, SUMMABILITY

9757:

Boyd, A. V. Summation of a double series. Math. Gaz. **44** (1960), 121–122.

Mordell [J. London Math. Soc. **33** (1958), 368–371; MR **20** #6615] has studied the summation of certain multiple series by means of the calculus. In the case of double series one of his results reduces to the following. If $T(c)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} 1/mn(m+n+c)$, then, for $c>-2$,

$$T(c) = 2\left\{1 - \frac{c-1}{1!2^3} + \frac{(c-1)(c-2)}{2!3^3} - \cdots\right\}.$$

The present author shows that if c is a positive integer, $T(c)$ may be evaluated by repeated application of the method of partial fractions.

V. F. Cowling (Lexington, Ky.)

9758:

Zmorovič, V. A. Sur le critère de V. P. Ermakov de convergence et divergence des séries à termes positifs. Bul. Inst. Politehn. Iaşi (N.S.) **5** (**9**) (1959), no. 3–4, 5–8. (Russian. French and Romanian summaries)

A single-valued function $f(x)$ is a function of the class PO if it is positive in $[1, \infty)$ and if it and its reciprocal are bounded on every finite subinterval of $(1, \infty)$; a single-valued function $\theta(x)$ is a function of the class E if it is positive and possesses a positive continuous derivative in $[1, \infty)$, and satisfies the inequality $\theta(x)>x$. Theorem: Let $f(x) \in PO$, $\theta(x) \in E$, and $\theta'(x) \to P$, when $x \to \infty$ where $1 \leq P < \infty$. If $f[\theta(x)]\theta'(x) \leq \alpha f(x)$, $0<\alpha<1$, the series $\sum_{n=1}^{\infty} f(n)$ converges, and if $f[\theta(x)]\theta'(x) \geq h f(x)$, $h>1$, the series diverges. The theorem can be considered as a complement to a theorem due A. Ostrowski [Enseignement Math. (2) **1** (1956), 224–257; MR **17**, 1076] and to the Ermakoff's test of convergence.

J. W. Andrushkiw (Newark, N.J.)

9759:

Zeller, Karl. ★Theorie der Limitierungsverfahren. Ergebnisse der Mathematik und ihrer Grenzgebiete. Neue Folge, Heft 15. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1958. viii+242 pp. DM 36.80.

[For an earlier review see MR **20** #4119.] The theory of summability, developed by many famous mathematicians, seems now to have reached a degree of completion. The time is ripe for a book which gives an encyclopedic exposition of the subject. In this task, the author is guided by what seems to be the last major trend of the theory, started by Banach, which consists in the exploitation of a topology, depending on the method A of summability used, in the set $\mathfrak{A}(A)$ of all A-summable sequences. Later the author replaced the normed topologies, used by the earlier writers, by arbitrary convex topologies (FK-spaces). Many important properties of the method A can be studied in terms of these topologies. This is the guiding point of view of the present book. However, some of the striking applications of the method, in particular the treatment of Tauberian theorems with gap conditions by Meyer-König and the author [Math. Z. **66** (1956), 203–224; MR **18**, 733] could not be explained in detail, since they were not yet completed when the book was in preparation. To some extent, the present book is complementary to the other excellent expositions of the theory, Hardy's *Divergent series* [Clarendon, Oxford, 1949; MR **11**, 25] and Pitt's *Tauberian theorems* [Oxford Univ. Press, London, 1958; MR **21** #5109]. While in the last two books the emphasis is on special summability methods, on sharp concrete theorems, Zeller's book stresses general summability methods, the description of the ideas and the tools of the proofs, and of the relations between different

groups of theorems. The headings of the eight chapters may serve as a description of the content: I. Fundamental notions of summability. II. Results from functional analysis. III. Structure of the summability fields. IV. Direct theorems of summability. V. Tauberian theorems. VI. Summability methods of Cesàro-Abel type (also Hölder, Riesz and Dirichlet means and the Laplace transformation are treated). VII. Methods of function theoretic type (the Nörlund, Euler, Borel methods, the "circle methods"). VIII. Further methods (Hausdorff, Rogosinski, Wiener means and others). To be able to cover this vast field, the author deliberately imposed some limitations on his book. Among other things, he does not cover applications or non-matrix methods. The style of the book is clear and readable, but condensed. At some places, one would wish to see more details in the explanations of ideas and proofs; at some others, selective remarks about the quoted papers. The bibliography at the end of the book (until 1955 inclusively) is very complete and contains some two thousand titles. We must be grateful to the author for this exposition of the subject, which will be found useful by all workers of this and adjacent fields and by those who wish to obtain a general impression of the subject. *G. G. Lorentz* (Syracuse, N.Y.)

9760:

Mehdi, M. R. Summability factors for generalized absolute summability. I. Proc. London Math. Soc. (3) **10** (1960), 180–200.

Avec les notations de Flett [mêmes Proc. (3) **7** (1957), 113–141; MR **19**, 266] l'auteur démontre, par exemple, qu'une condition nécessaire et suffisante pour que $\sum \varepsilon_n a_n$ soit sommable $|A|_k$ $(k \geq 1)$ lorsque $\sum a_n$ est sommable $|C, \alpha|$ est que $\Delta^\alpha \varepsilon_n = O(n^{-\alpha})$, $\varepsilon_n = O(1)$ pour tout $\alpha \geq 0$ réel. *S. Mandelbrojt* (Paris)

9761:

Glatfeld, Martin. On strong Rieszian summability. Proc. Glasgow Math. Assoc. **3**, 123–131 (1957).

Eine Reihe $\sum_1^\infty c_n$ heißt $|R, \lambda, k|^p$-summierbar zum Werte c: $\sum_1^\infty c_n = c\,|R, \lambda, k|^p$, wo $0 < p < \infty$, $k > -1/p$, $0 \leq \lambda_n \nearrow +\infty$, wenn eine Zahl c existiert, so daß

$$\int_0^\infty |C_\lambda^k(\tau) - c\tau^k|^p \, d\tau = o(\omega^{kp+1}), \quad C_\lambda^k(\tau) = \sum_{\lambda_n < \tau} c_n(\tau - \lambda_n)^k,$$

[vgl. Richert, Nachr. Akad. Wiss. Göttingen, Math.-Phys. Kl. IIa **1956**, 77–125; MR **18**, 123]. Verf. beweist eine Reihe von Relationen zwischen verschiedenen Verfahren dieser starken Rieszschen Summierbarkeit sowie ihre Beziehungen zur gewöhnlichen und absoluten Rieszschen Summierbarkeit. Beispiele: (1) Aus $|R, \lambda, k|^p$ folgt $|R, \lambda, k|^{p'}$ für $0 < p' < p$. (2) Aus $|R, \lambda, k|^p$ folgt $|R, \lambda, k'|^p$ für $p \geq 1$, $k' > k$. (3) Aus $|R, \lambda, k|^p$ folgt $|R, \lambda, k + 1/p - 1/p'|^{p'}$ für $1 < p < p' < \infty$. Für das Dirichletsche Produkt zweier Reihen, die $|R, \lambda, k|^p$- bzw. $|R, \rho, \mu|^{p'}$-summierbar sind $(1/p + 1/p' \leq 1)$ beweist Verf. gewöhnliche $(R, \nu, k + \mu + 1)$-Summierbarkeit. Für $p \geq 1$, $kp > -1$ wird gezeigt, daß das spezielle $|R, n, k|^p$-Verfahren mit der von Boyd und Hyslop [Proc. Glasgow Math. Assoc. **1** (1952), 94–99; MR **14**, 463] definierten starken Rieszschen Summierbarkeit äquivalent ist. Ein zu $|R, \lambda, k|^p$ äquivalentes Verfahren wurde unabhängig von Pramila Srivastava [Proc. Nat. Inst. Sci. India Part A **23** (1957), 58–71; MR **20** #2555] eingeführt. *H.-E. Richert* (Göttingen)

9762:

Burkill, H. On Riesz and Riemann summability. Proc. Cambridge Philos. Soc. **57** (1961), 55–60.

The author introduces a simple generalization of Cesàro's method as follows: If k is a positive integer, $0 \leq \lambda_0 < \lambda_1 < \cdots$ and

$$t_n = \sum_{\nu=0}^{\infty} \left(1 - \frac{\lambda_\nu}{\lambda_{n+1}}\right) \cdots \left(1 - \frac{\lambda_\nu}{\lambda_{n+k}}\right) a_\nu \to t$$

as $n \to \infty$, then $\sum_{n=0}^\infty a_n$ is said to be summable (C, λ_n, k) to t. It is proved that if (*) $0 < p \leq \lambda_{n+1} - \lambda_n \leq q$, then the method (C, λ_n, k) is equivalent to the Riesz typical mean (R, λ_n, k). Using this equivalence, the author proves the following theorem: If $\sum_{n=0}^\infty a_n$ is summable (R, λ_n, k) to s, where k is a positive integer, $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots$, and if (*) holds, then it is summable to s also by the "generalized Riemann method" (R, l, λ_n) defined by $\tau_h = a_0 + \sum_{n=1}^\infty (\sin \lambda_n h / \lambda_n h)^l a_n \to s$ as $h \to 0$, where l is a positive rational number with odd denominator and $l > k + 1$. This is similar to a result of Verblunsky's [same Proc. **26** (1930), 34–42] on Cesàro and Riemann methods of summability. *M. R. Parameswaran* (Madras)

9763:

Albrycht, J.; Radecki, J. On the summability of generalized Bernstein-Chlodovsky polynomials. Bull. Soc. Amis Sci. Lettres Poznań. Sér. B **15** (1958/59), 87–97 (1960).

For the polynomials $B_n(x) = \sum_{k=0}^n \binom{n}{k} u^k (1-u)^{n-k} M_{n,k}$, where $u = x/h_n$ and $M_{n,k}$ is the arithmetic mean of the function $f(t)$ over the interval $(k(n+1)^{-1}h_n, (k+1)(n+1)^{-1}h_n)$, the authors prove (a) $nh_n^{-1}(B_n(x) - f(x)) \to \frac{1}{2}(f'(x) + xf''(x))$; (b) $B_n(x)$ is $C_{-1+\delta}$ summable to $f(x)$. The following assumptions are made: $f''(x)$ exists, $h_n \to \infty$, $0 < h_n \leq \text{const}\ n^\alpha$, $0 < \alpha < 1$, $\delta > \alpha$ and the maximum of $|f(t)|$ on $(0, h_n)$ is $o(n^{-1}h_n \exp(\beta n/h_n))$, $\beta > 0$.

G. G. Lorentz (Syracuse, N.Y.)

9764:

Gould, H. W. A q-binomial coefficient series transform. Bull. Amer. Math. Soc. **66** (1960), 383–387.

Let

$$(*) \quad F(n) = q^{n(n-1)/2} \sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} a+bk \\ n \end{bmatrix} q^{k(k-n)} f(k),$$

where $\begin{bmatrix} a \\ n \end{bmatrix} = \prod_{j=1}^n (q^{a-j+1} - 1)/(q^j - 1)$. It is proved that if a and b are non-negative integers then (*) implies

$$\sum_{n=0}^\infty (-1)^n F(n) u^n = \sum_{k=0}^\infty \begin{bmatrix} a+bk \\ k \end{bmatrix} q^{k(k-1)/2} u^k f(k) \times \prod_{j=0}^{a+bk-k-1} (1 - uq^j).$$

In the limiting case $q = 1$, (*) becomes

$$(**) \qquad F(n) = \sum_{k=0}^n (-1)^n \binom{n}{k} \binom{a+bk}{n} f(k).$$

It is shown that (**) implies

$$\binom{a+bn}{n} f(n) = \sum_{k=0}^n (-1)^k \frac{a+bk-k}{a+bn-k} \binom{a+bn-k}{n-k} F(k).$$

Also the Abel transform

$$F(n) = \sum_{k=0}^{\infty} (-1)^k \binom{n}{k} \frac{(a+bk)^n}{n!} f(k)$$

is discussed and an explicit formula for the inverse is obtained. *L. Carlitz* (Durham, N.C.)

9765:

Borozdin, K. V. Generalized Abel theorems. Dokl. Akad. Nauk SSSR **137** (1961), 1270–1273 (Russian); translated as Soviet Math. Dokl. **2**, 441–444.

Two theorems are proved. The first states that if $xf(x)$ is continuous and of bounded variation in $[0, 1]$ and if $\sum_{i=0}^{\infty} a_i = A$, then $\lim_{x\to 1-0} \sum_{i=0}^{\infty} a_i x^i f(x^i) = Af(1)$. The second states that if $\sum_{k=0}^{\infty} a_k \sim n^\alpha/\Gamma(1+\alpha)$ for $n\to\infty$ with $\alpha > 0$ and $x^\beta f(x)$ is continuous and of bounded variation in $[0, 1]$ for some $\beta \in [0, 1]$, then

$$\lim_{x\to 1-0} (1-x)^\alpha \sum_{k=0}^{\infty} a_k x^k f(x^k) = \frac{1}{\Gamma(\alpha)} \int_0^1 f(x) \left(\ln \frac{1}{x}\right)^{\alpha-1} dx.$$

F. Goodspeed (Quebec)

9766:

Kosambi, D. D. Classical Tauberian theorems. J. Indian Soc. Agric. Statist. **10** (1958), 141–149.

The author uses arguments of probability theory to motivate the classical Tauberian conditions such as occur in Littlewood's theorem for power series and the Tauberian theorem for Borel summability. He considers summability methods of the form $s(t) = \sum b_n s_n t^n / \sum b_n t^n$ where $b_n > 0$, $\varphi(t) = \sum b_n t^n$ converges for $|t| < t^*$, and $b_n t^n/\varphi(t) \to 0$ as $t \uparrow t^*$. He now introduces the stochastic variable X_t which takes on the value n with probability $b_n t^n/\varphi(t)$. Setting $x = \{n - \mu(t)\}/\sigma(t)$ where μ and σ^2 denote the expectation and variance of X and letting $t \uparrow t^*$ he obtains a certain limiting distribution. The scale factor in the transformation from t to x, evaluated at $x = 0$, determines the form of the Tauberian condition. Case (i): If φ is an entire function of finite order the limiting distribution is Gaussian and the Tauberian condition must be $a_n = O(n^{-1/2})$. Case (ii): If the power series for φ has finite positive radius of convergence the author obtains the Tauberian condition $a_n = O(n^{-1})$. The proofs make use of uniformly equicontinuous families of functions derived from the partial sums s_n. {Reviewer's remark: This tool has been used before in the proof of general Tauberian theorems, notably by Beurling [Acta Math. **77** (1945), 127–136; MR 7, 61].} *J. Korevaar* (Madison, Wis.)

9767:

Meyer-König, W.; Zeller, K. Ein high-indices-Theorem für Dirichletsche Reihen. Math. Z. **74** (1960), 221–231.

The following high indices theorem for Dirichlet series $\sum u_n e^{-\lambda_n z}$, $0 \le \lambda_0 < \lambda_1 < \cdots$, $\lambda_n \to \infty$, is proved. If $\lambda_{n+1} - \lambda_n \ge$ const $\sqrt{\lambda_n}$, if $u(z) = \sum u_n e^{-\lambda_n z}$ converges for $\operatorname{Re} z > 0$, and if the Taylor series of $u(z)$ about some point $a > 0$ converges for $z = 0$, then $\sum u_n$ converges. For comparison, in the theorem of Hardy and Littlewood [Hardy, *Divergent series*, Clarendon Press, Oxford, 1949; MR **11**, 25; p. 172] the conclusion is the same, while the conditions are $\lambda_{n+1} - \lambda_n \ge$ const λ_n and the Abel summability of the Taylor series at $z = 0$. The proof is an application of the general methods developed by the authors [Math. Z. **66** (1956), 203–224; MR **18**, 733] for Tauberian theorems with gap conditions. *G. G. Lorentz* (Syracuse, N.Y.)

9768:

Meynart, C. ★Les séries et leur application à la résolution de divers problèmes pratiques d'analyse mathématique. Tome 2. Eyrolles, Paris, 1960. 116 pp. (10 inserts) 28.00 NF.

Volume 2 continues the pattern of volume 1 [Eyrolles, Paris, 1959; MR **22** #1779] in presenting a number of examples of the solution of various mathematical problems by means of infinite series. As in volume 1 the work is purely formal with no discussion of convergence or the limitations of the method.

Chapter VII gives examples of series solutions of differential systems. Chapter VIII deals with operational calculus and its relation to series. Chapter IX discusses several practical problems including examples from electricity and ballistics. Chapter X contains some brief remarks on high-speed automatic computers, both digital and analogue. *J. G. Herriot* (Stanford, Calif.)

APPROXIMATIONS AND EXPANSIONS

See also 9738, 9763, 9784, 9793, 9816a–b.

9769:

Suetin, P. K. Interpolation of analytic functions. Dokl. Akad. Nauk SSSR **128** (1959), 1125–1128. (Russian)

Let G be a bounded simply connected domain bounded by a simple curve Γ. Let $g(z)$ be analytic in the exterior D of Γ and continuous and non-vanishing in $\bar{D}$. The author investigates the degree of approximation of a function $f(z)$ analytic in G by interpolation polynomials $L_n(z)$ where the points of interpolation are the zeros of the $(n+1)$st Faber polynomial of G with weight $g(z)$. A typical result: If $f(z)$ has k derivatives in $\bar{G}$ and the kth derivative satisfies a Lipschitz condition of order β, if $g(z)$ has two continuous derivatives in $\bar{D}$, and if the mapping function $\psi: |\omega| > 1 \to D$ ($\psi(\infty) = \infty$) has four continuous derivatives in $|\omega| \ge 1$, then $|f(z) - L_n(z)| < c_1 n^{-k-\beta} \log n$ ($z \in \bar{G}$). *W. H. J. Fuchs* (Ithaca, N.Y.)

9770:

Walsh, J. L. ★Approximation by bounded analytic functions. Mémor. Sci. Math., Fasc. 144. Gauthier-Villars, Paris, 1960. 66 pp. 13 NF; $2.85.

Much of the author's recent work has dealt with the subject mentioned in the title. More explicitly, given a complex-valued function $f(z)$ on a closed set E of the complex plane and given a domain D containing E, one seeks information on how well $f(z)$ may be approximated on E by functions $\varphi_M(z)$ analytic in D and bounded in modulus by M there. Of special interest is the asymptotic behavior of the degree of approximation as $M \to \infty$. As the author points out in his preface, such questions not only apply to polynomial and rational function approximation, but also are significant whenever a sequence of functions analytic in a region converge in a subregion.

The present book is the first treatise on approximation by bounded analytic functions and is based mainly on

several of the author's papers. As the author admits, there are still open problems. Indeed, some of them are presented in the final chapter, thus enhancing the value of the book for anybody who wants to do research in this area. Nevertheless, the book is written so as to be readable by the non-specialist and requires only a basic knowledge of function theory.

Topics covered include problem A in which $f(z)$ is analytic on E, problem α in which $f(z)$ has properties weaker than analyticity on E (e.g., Lipschitz condition), and problem β in which $f(z)$ is analytic on E but has weaker properties on a closed set E_1 containing E. The author also indicates how the theory may be modified so as to apply to questions of extremal interpolation involving norms not necessarily Čebyšev.

D. S. Greenstein (Evanston, Ill.)

9771:

Maehly, H.; Witzgall, Ch. Tschebyscheff-Approximationen in kleinen Intervallen. I. Approximation durch Polynome. Numer. Math. **2** (1960), 142–150.

Let f be continuous in $[-b, b]$ and P the polynomial of degree n which best approximates f in the sense of minimizing $\|(P-f)v^{-1}\| = \max_{[-b,b]}|(P-f)v^{-1}|$ with v a continuous positive weight function. The authors are concerned with the behavior of P as $b \to 0$. Denote the "error function" by $\delta(x) = \delta(f, v) = (P-f)v^{-1}$. They first prove that P depends continuously on f and v. More precisely, let f and g be continuous in $[-b, b]$, P and U the nth degree polynomials of best approximation to f and g, relative to weight functions v and w, respectively. Then, if $|\|v\|\|v^{-1}\| - \|w\|\|w^{-1}\|| \leq 1$, there is a constant K depending only on f, v, n and b such that $\|P-U\| \leq K(\|f-g\| + |\lambda|\|v-w\|)$ where $|\lambda| = \|\delta(f, v)\|$. In a note added in proof, reference is made to G. Freud [Acta Sci. Math. Szeged **19** (1958), 162–164; MR **21** #251] for a similar theorem. As is well known, there are at least $n+2$ consecutive points $x_k \in [-b, b]$, here called critical points of f relative to v, where the error function $\delta(f, v)$ assumes the value $\|\delta\|$ with alternating signs. Let x_0 be the smallest x for which $|\delta(x)| = \|\delta\|$, and denote $\delta(x_0) = \lambda(f, v)$. It is shown that also λ depends continuously on f and v provided that the critical points are uniquely determined. These results are used to prove the following theorem. Let f and v be analytic functions in a neighborhood of the origin and $F_b(y) = f(by)$, $V_b(y) = v(by)$, $\delta_b(y) = \delta(F_b, V_b)$, $\lambda_b = \lambda(F_b, V_b)$ ($|y| \leq 1$). Then

$$\lim_{b \to 0} \delta_b(y)/\lambda_b = (-1)^{n+1} T_{n+1}(y)$$

uniformly in $[-1, 1]$, where T_{n+1} is the Čebyšev-polynomial of degree $n+1$. Also,

$$\lim \lambda_b / b^{n+1} = (-1)^n 2^{-n} f^{(n+1)}(0)/(n+1)!$$

as $b \to 0$. {Reviewer's remark: The assumption of analyticity could be relaxed to that of the existence of the $(n+1)$st derivative at the origin, for f, and continuity at the origin, for v.} *Walter Gautschi* (Oak Ridge, Tenn.)

9772:

Szász, Paul. Remarque sur un ouvrage de M. Léopold Fejér. Ann. Univ. Sci. Budapest. Eötvös. Sect. Math. **2** (1959), 151–155.

Theorem: Let $a \leq \eta_0 < \xi_1 < \eta_1 < \xi_2 < \cdots < \xi_n < \eta_n \leq b$, $f(x)$ continuous in $[a, b]$ and $F_{2n-1}(x)$ a polynomial of degree $2n-1$ at most, such that

$$F_{2n-1}(\xi_k) = f(\xi_k) \quad (k = 1, 2, \cdots, n)$$

and

$$\max_{x \in [a,b]} |f(x) - F_{2n-1}(x)| = f(\eta_k) - F_{2n-1}(\eta_k) \quad (k = 0, 1, \cdots, n).$$

Then for every polynomial $g_{2n-1}(x) \not\equiv F_{2n-1}(x)$ of degree $2n-1$ at most satisfying $g_{2n-1}(\xi_k) = f(\xi_k)$ $(k = 1, 2, \cdots, n)$ we have

$$\max_{x \in [a,b]} |f(x) - g_{2n-1}(x)| > \max_{x \in [a,b]} |f(x) - F_{2n-1}(x)|.$$

Application: Let us denote by π_n an arbitrary polynomial of degree n at most; it is a well-known result of L. Fejér [Amer. Math. Monthly **41** (1931), 1–14] that for a normally distributed point system $\{\xi_k\}$,

$$\min_{\pi_{2n-1}(\xi_k) = f(\xi_k)} \max_{x \in [a,b]} |f(x) - \pi_{2n-1}(x)| \leq$$

(*)

$$2 \min_{\pi_{2n-1} \text{ arbitrary}} \max_{x \in [a,b]} |f(x) - \pi_{2n-1}(x)|.$$

By means of the theorem above it is shown that in relation (*) the sign of equality is valid for $f(x) = x^{2n}$ $a = -1$, $b = +1$ and $\xi_k = \cos(2k-1)\pi/2n$ $(n = 1, 2, \cdots)$.

G. Freud (Budapest)

9773:

Walsh, J. L.; Motzkin, T. S. Polynomials of best approximation on an interval. Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 1523–1528.

The authors extend their study of juxtapolynomials from finite point sets [Trans. Amer. Math. Soc. **91** (1959), 231–245; MR **21** #7388] to closed finite intervals E. If p_n and q_n are two polynomials of degree n, q_n is said to be closer to f than p_n if $|f-q_n| < |f-p_n|$ on E, if $f-p_n \neq 0$, and $p_n = q_n$ otherwise. Then p_n is a juxtapolynomial to f if no closer polynomial of degree n exists. Let the upper multiplicity of a zero x_0 of a continuous function $\varphi(x)$ be defined as the smallest integer μ such that for $x \in E \cap \{0 < |x-x_0| < \delta\}$ and some m, $|\varphi(x)| > |m(x-x_0)^\mu| > 0$ and $m\varphi(x)(x-x_0)^\mu > 0$. Theorem: $p_n(x)$ of degree n is a juxtapolynomial to the continuous function $f(x)$ on E if and only if $f-p_n$ has on E zeros of total upper multiplicity greater than n. Of particular interest are juxtapolynomials which are best approximants to f in L^p norm. A new and simpler proof is given for a theorem of Jackson to the effect that if p_n is a best approximating polynomial to a continuous f in L^1 norm on $(0, 1)$, then $f-p_n$ either has a total of $n+1$ strong changes of sign, or vanishes identically on a set of positive measure. In the latter case, a lower bound is given for the measure of such a subset.

H. O. Pollak (Murray Hill, N.J.)

9774:

Collatz, Lothar. Tschebyscheffsche Annäherung mit rationalen Funktionen. Abh. Math. Sem. Univ. Hamburg **24** (1960), 70–78.

Continuing his earlier work on Čebyšev approximation of functions of several variables [in particular (*) Z. Angew. Math. Mech. **36** (1956), 198–211; MR **18**, 677], the author considers approximation of a continuous functions $g(x)$, $x = (x_1, x_2, \cdots, x_k)$ in a closed domain B by functions of

the form $Q=\sum_0^n a_\nu u_\nu(x)/\sum_0^m b_\mu u_\mu(x)$, where n and m are fixed and the u_ν are continuous. Assuming the existence of Q_0 of the above form such that $|Q_0-g|\leq M$ for $x\in B$, and also the existence of subsets M_1 and M_2 of B such that $Q-Q'$ of opposite sign on M_1 and M_2 is impossible for no Q and Q' of the above form, the author proves the same estimate for $\min|Q-g|$ as in (*). The theory is applied to obtain known results on rational approximation in one variable as well as new results on approximation by the ratio of two linear functions in K variables. A very interesting series of illustrations shows the many possible forms for the set on which the Čebyšev approximation error may assume its extreme value. It is also shown that in distinction to the results of (*), smoothness conditions on g and its first derivatives as well as strong convexity of B no longer suffice for uniqueness of the optimal approximating function.

{In (19) and the line preceding, read "$n+m+2$" for "$n+m-2$".} *H. O. Pollak* (Murray Hill, N.J.)

9775:

Larcher, Heinrich. Notes on orthogonal polynomials in two variables. Proc. Amer. Math. Soc. **10** (1959), 417–423.

Let B be a plane region $\Phi_1(y)\leq x\leq\Phi_2(y)$, $a\leq y\leq b$, where Φ_1 and Φ_2 are both of the form $cy+d\pm(ey^2+fy+g)^{1/2}$. The problem is to find the system of polynomials $p_{mn}(x, y)$ orthogonal in B, where p_{mn} is of total degree $m+n$ in x and y, the coefficient of x^my^n does not vanish, and p_{mn} may contain terms x^sy^t for $s+t<m+n$, and $s+t=m+n$ if $s<m$.

Under the given conditions on Φ_i, $p_{mn}(x, y)$ may be written as $p_n(x, y)p_n(y)$, where $p_n(x, y)$ are Legendre polynomials in x defined in the interval $[\Phi_1(y), \Phi_2(y)]$, and $p_n(y)$ is the polynomial of degree n in the system orthogonal in $[a, b]$ under the weight function

$$w_m(y)=\int_{\Phi_1(y)}^{\Phi_2(y)}[p_m(x, y)]^2\,dx.$$

The author then establishes a variational problem for $p_n(y)$ with the aid of a Lagrange multiplier and solves this problem explicitly for the case of ellipses, parabolic regions, and certain parallelograms and triangles.

H. O. Pollak (Murray Hill, N.J.)

9776:

Džrbašyan, M. M.; Nersesyan, A. B. Expansion in special biorthogonal systems and boundary value problems for differential equations of fractional order. Dokl. Akad. Nauk SSSR **132** (1960), 747–750 (Russian); translated as Soviet Math. Dokl. **1**, 629–633.

The authors define two systems of functions, $\{Y_k(x, \rho)\}$, $\{Z_k(x, \rho)\}$ which are biorthogonal on $[0, l]$. If $f\in L_1(0, l)$ and is bounded in some neighborhoods of 0 and l, then with suitable conditions on the parameters involved in the construction of Y_k and Z_k, $\sum_1^\infty(\int_0^l f(t)Z_k(t, \frac{1}{2})dt)Y_k(x, \frac{1}{2})$ is uniformly equiconvergent in $[\varepsilon, l-\varepsilon]$ with the Fourier series of f to difference 0. An analogous theorem holds for the system $\{Z_k\}$.

For $\rho>\frac{1}{2}$ similar results are given under the hypothesis that f is the αth fractional integral of a function of L_1 which is bounded in small neighborhoods of 0 and l. α here depends on ρ and the parameters previously mentioned. Terms Y_0 and Z_0 are added to the sequences. A condition is given under which the equiconvergence is to difference 0 and another under which the difference is the product of a function of ρ and the first term of the authors' development.

The authors then construct a set of "mutually adjoint" boundary-value problems, involving fractional derivatives, for which the given systems of functions are the eigenfunctions. *D. Waterman* (Detroit, Mich.)

9777:

Schäfke, Friedrich Wilhelm. Reihenentwicklungen analytischer Funktionen nach Biorthogonalsystemen spezieller Funktionen. I. Math. Z. **74** (1960), 436–470.

The author summarizes and supplies details of proof of certain results published previously elsewhere [J. Meixner and F. W. Schäfke, *Mathieusche Funktionen und Sphäroidfunktionen mit Anwendungen auf physikalische und technische Probleme*, Springer, Berlin, 1954; MR **16**, 586; pp. 82–89 and 128]. These results are now applied to Hill functions. Part II, not yet available, will be concerned with expansion theorems related to Bessel, Legendre and Whittaker functions. *C. J. Bouwkamp* (Eindhoven)

9778:

R.-Salinas, Baltasar. Equivalenza di classi di funzioni con sviluppo asintotico in un angolo. Funzioni caratteristiche. Boll. Un. Mat. Ital. (3) **14** (1959), 525–531. (English summary)

Soit Ω un domaine contenu dans un angle $|\arg z|<\alpha\pi/2$ de la surface de Riemann de $\log z$, $z=0$ étant un point de la frontière de Ω. Désignons par $K_0\{m_n; \Omega\}$ la classe des fonctions analytiques dans Ω telles qu'il existe une série $\sum a_nz^n$ avec $|f(z)-\sum_0^{n-1}a_\nu z^\nu|\leq Aq^nm_n|z^n|$ $(n\geq 1;$ $z\in\Omega)$. Soit A_α l'angle d'ouverture $\alpha\pi\leq 2\pi$. Soit $0<\alpha\leq 2$. Si $\lambda=\liminf m^{1/n}/n^{\alpha-2}=0$, $K_0\{m_n; A_\alpha\}$ ne contient que des constantes; si $0<\lambda<\infty$, les classes $K_0\{m_n; A_\alpha\}$ et $K_0\{n^{(\alpha-2)n}; A_\alpha\}$ coïncident. $\{m_n^{(\alpha)}\}$ est défini par le fait que $\{\log[n^{(2-\alpha)n}m_n^{(\alpha)}]\}$ est la régularisée convexe de $\{\log[n^{(2-\alpha)n}m_n]\}$. Si $\lambda>0$, les classes $K_0\{m_n; A_\alpha\}$ et $K_0\{m_n^{(\alpha)}; A_\alpha\}$ sont équivalentes. La condition

$$(m_n'^{(\alpha)}/m_n'')^{1/n}<M<\infty$$

est nécessaire et suffisante pour que $K_0\{m_n'; A_\alpha\}\subset K_0\{m_n''; A_\alpha\}$. *S. Mandelbrojt* (Paris)

9779:

Hsu, L. C. Some approximation formulas for the integration of violently oscillating functions and of periodic functions. Sci. Record (N.S.) **3** (1959), 544–549.

This paper gives a number of theorems concerning the integration of continuous functions of several variables, which are periodic in each of the variables except perhaps one.

The first part considers integrals of $\Phi(x, y, \cdots, y_k)$, periodic in the y_r, over a range of x with $y_r=N_rx$, where the $N_r\to\infty$ in such a manner that $N_{r+1}/N_r\to\infty$ also. Theorem 1 gives a limiting value as $N_1\to\infty$ (and hence all $N_r\to\infty$, $r\geq 1$). Theorems 2 and 3 give an error bound and an expression for the error when N_1 is large in the case $k=1$, i.e., that of functions of two variables.

The second part of the paper is concerned with the integration of $f(x_1, x_2, \cdots, x_n)$ over the period-interval $(0, 2\pi)$ in each of its variables. Theorem 4 expresses the

result in terms of a single integral with respect to t, and an error term depending on $1/R$, of an integrand $\Phi(t)=f(R^{\alpha_1}t, R^{\alpha_2}t, \cdots, R^{\alpha_n}t)$, where $\alpha_1>\alpha_2>\cdots>\alpha_n$ and we suppose R is large.

Theorem 5 expresses the same integral, again with an error term, and using a similar device, as a finite sum (but with very many terms).

J. C. P. Miller (Cambridge, England)

9780:

Balk, M. B. The Padé interpolation process for certain analytic functions. Issledovaniya po sovremennym problemam teorii funkcii kompleksnogo peremennogo, pp. 234–257. Gosudarstv. Izdat. Fiz.-Mat. Lit., Moscow, 1960. (Russian)

The author considers various definitions and general properties of infinite tables $\{a_{\mu\nu}\}$ where the row-index μ and the column-index ν vary from 0 to $+\infty$. The definitions are applied to the study of rational functions which Padé has associated with the Taylor expansion $\sum c_n z^n = f(z)$. The author introduces special classes of functions (in terms of the behavior of the coefficients c_n) and studies the convergence of the associated Padé tables. The following functions are among those investigated: $\sum_{n=0}^{\infty} b^{n^2}z^n$ $(0<|b|<1)$, $a_0+\sum_{n=1}^{\infty} z^n/\Gamma(a+n+1)$, $\prod_{n=1}^{\infty}(1-q^n z)$ $(0<|q|<1)$, $A+\sum_{n=1}^{\infty} q^n(1+q^n)^{-1}z^n$ $(0<|q|<1)$.

A. Edrei (Syracuse, N.Y.)

FOURIER ANALYSIS

See also 9575, 9701, 9823.

9781:

Matthies, K.; Mazkewitsch, D. Developments of $\sin^\alpha x$ into Fourier series. Amer. Math. Monthly **67** (1960), 52–54.

The authors find by straightforward calculation using Moivre's formulas and the Beta function: For all x on the interval $k\pi<x<(k+1)\pi$ where $k=0, \pm1, \pm2, \cdots$, and all real numbers a we have,

$$(-1)^k|\sin x|^a = \frac{2}{\sqrt{\pi}}\frac{\Gamma(1+a/2)}{\Gamma(1-(a/2+1/2))} \times \sum_{n=0}^{\infty}\frac{\Gamma(n+1-(a/2+1/2))}{\Gamma(n+1+(a/2+1/2))}\sin(2n+1)x,$$

and

$$|\sin x|^a = \frac{2}{\sqrt{\pi}}\frac{\Gamma(a/2+1/2)}{\Gamma(-a/2)} \times \left[\frac{\Gamma(-a/2)}{2\Gamma(1+a/2)}+\sum_{n=1}^{\infty}\frac{\Gamma(n-a/2)}{\Gamma(n+1+a/2)}\right]\cos 2nx.$$

G. Goes (Evanston, Ill.)

9782:

van Albada, P. J.; Laman, G.; van Lint, J. H. On the hermitean product of ordered point sets on the unit circle. Michigan Math. J. **7** (1960), 245–250.

The following was a conjecture of C. J. Titus and J. L. Ullman (communicated to one of the authors by way of N. Kazarinoff, the reviewer and J. Seidel): If $0\le\phi_j\le 2\pi$ and $0\le\psi_j<2\pi$ for $1\le j\le n$, $\phi_j<\phi_{j+1}$ and $\psi_j<\psi_{j+1}$ for $1\le j\le n-1$, and $\sum_{j=1}^{n} e^{i\phi_j}=0=\sum_{j=1}^{n} e^{i\psi_j}$, then

$$\sum_{j=1}^{n} e^{i(\phi_j-\psi_j)} \neq 0.$$

The authors prove the conjecture and establish an analogue for functions on the unit circle which are one-sidedly continuous. They prove that the greatest lower bound of the sum is 0 if and only if n is divisible by 4. When $n\not\equiv 0 \pmod 4$, they compute lower bounds for the sum and make a conjecture as to the greatest lower bound.

Victor Klee (Seattle, Wash.)

9783:

Rosenblum, Marvin; Widom, Harold. Two extremal problems. Pacific J. Math. **10** (1960), 1409–1418.

Let P be any complex trigonometric polynomial of the form $P_0+P_1e^{i\phi}+P_2e^{2i\phi}+\cdots$. Let σ and μ denote, respectively, normalized Lebesgue measure and any finite non-negative Borel measure on the real interval $(-\pi, \pi]$. The authors solve the following extremal problems. (1) Given g such that $\int|g|^2d\mu<\infty$, find $\inf[\int|g+P|^2d\mu]^{1/2}$. (2) Given h such that $\int|h|d\sigma<\infty$, find $\sup\{|\int Ph\,d\sigma|/[\int|P|^2d\mu]^{1/2}\}$. Various problems concerning Toeplitz forms are special cases of these problems. The authors' solutions to both problems depend on the solution of a single type of Riemann-Hilbert problem, which can be handled by singular integral equation techniques.

D. S. Greenstein (Evanston, Ill.)

9784:

Golinskiĭ, B. L. Local approximation of two conjugate functions by trigonometric polynomials. Mat. Sb. (N.S.) **51 (93)** (1960), 401–426. (Russian)

The author proves local analogues for several known results on Fourier series. For example (corresponding to the well-known theorem of M. Riesz), if $f\in L'[0, 2\pi]\cap L^p[a, b]$ $(p>1)$ where $a, b\in[0, 2\pi]$ and $b-a<2\pi$, then the conjugate function $\bar{f}$ is in $L^p[a', b']$ for any interval $[a', b']$ interior to $[a, b]$. Almost all the results involve too much notation to reproduce here, but many of them are interesting. Among these results are theorems on local approximation of f and $\bar{f}$ by Fejer and Jackson sums, and conditions in terms of moduli of continuity for convergence almost everywhere on $[a, b]$ of the Fourier series and conjugate series of a function in $L'[0, 2\pi]\cap L^p[a, b]$.

R. R. Goldberg (Evanston, Ill.)

9785:

Nikol'skiĭ, S. M. Uniform differential properties of an analytic function in a strip. Mathematica (Cluj) **2 (25)** (1960), no. 1, 149–157. (Russian)

If $F(x)$ is a real function of period 2π, let $E_n(F)_p$ be the distance of best approximation to $F(x)$ by trigonometric polynomials of degree n in the metric of L^p. Let δ and τ be fixed positive numbers. The problem is to find conditions on $F(x)$ that $E_n(F)_p=O(e^{-\delta n}n^{-\tau})$. If $F(x)$ is redefined on a set of measure zero, it is the restriction of a function $F(x+iy)$, defined and analytic for $|y|<\delta$, and

$$\lim_{y\to-1}\operatorname{Re}F(x+iy)=\lim_{y\to+1}\operatorname{Re}F(x+iy).$$

If the limit is denoted by $\varphi(x)$, it is periodic of period

2π and represents $F(x+iy)$ as an integral by a variation of Cauchy's formula. The desired approximation is related to the existence of τ derivatives for $\varphi(x)$ in L^p, with a suitable interpretation of "derivative" by bounds on difference quotients when τ is not an integer.

L. de Branges (Bryn Mawr, Pa.)

9786:

Tomić, M. A note on lacunary Fourier series. Proc. Amer. Math. Soc. **11** (1960), 460–462.

The author proves the following theorem. Let $f(x)$ be continuous at $x=x_0$, and let the Fourier series of $f(x)$ have finitely many gaps (n_ν, n_ν'), where $n_\nu'-n_\nu\to\infty$ and $n_\nu'/n_\nu\to 1$. Suppose that

$$\omega\left(x_0, \frac{\pi}{n_\nu'-n_\nu}\right)\log\left(1-\frac{n_\nu}{n_\nu'}\right)\to 0,$$

where $\omega(x_0,\delta)=\sup_{0<|t|\le\delta}\{|f(x_0+t)-f(x_0)|\}$. Then

$$S_{n_\nu}(x_0)\to f(x_0),$$

where $S_n(x)$ denotes the partial sum of the Fourier expansion of f at x_0.

E. M. Stein (Chicago, Ill.)

9787:

Taĭkov, L. V. Divergence of Fourier series. Dokl. Akad. Nauk SSSR **137** (1961), 782–785 (Russian); translated as Soviet Math. Dokl. **2**, 347–350.

Theorem: If E is a set of type F_σ in the interval $[0, 2\pi)$, there exist two conjugate Fourier series which converge on E and diverge unboundedly on the complement of E. The author's construction is based on the Kolmogorov example and on the method of Herzog and Piranian [Duke Math. J. **16** (1949), 529–534; MR **11**, 91] and Zeller [Arch. Math. **6** (1955), 335–340; MR **16**, 1015]. The reviewer is unable to verify all the technical details.

G. Piranian (Ann Arbor, Mich.)

9788:

Katayama, Miyoko. Fourier series. VII. Uniform convergence factors of Fourier series. J. Fac. Sci. Hokkaido Univ. Ser. I **13** (1957), 121–129.

Let

$$f(x)\sim \tfrac{1}{2}a_0+\sum a_\nu\cos\nu x+b_\nu\sin\nu x$$

and let $\{\lambda_\nu\}$ be a sequence of real numbers. If $\frac{1}{2}a_0\lambda_0+\sum\lambda_\nu(a_\nu\cos\nu x+b_\nu\sin\nu x)$ converges uniformly, then $\{\lambda_\nu\}$ is called a sequence of uniform convergence factors of the Fourier series of f. Let $K_n=\frac{1}{2}\lambda_0+\sum_1^n\lambda_\nu\cos\nu t$. Karamata [J. Math. Pures Appl. (9) **35** (1956), 87–95; MR **17**, 964] showed that a necessary and sufficient condition that $\{\lambda_\nu\}$ have this property for the class of continuous f is $\int_0^\pi|K_n(t)|dt=O(1)$. The author gives necessary and sufficient conditions for other classes as follows: (a) Bounded measurable; $\int_{-\pi}^{\pi}|K_n(t)|dt=O(1)$ and $\int_H|K_m(t)-K_n(t)|dt<\varepsilon$ for any set H of measure $<\eta=\eta(\varepsilon)$. (b) L_p $(p>1)$; $\int_{-\pi}^{\pi}|K_n(t)|^q dt=O(1)$, $1/p+1/q=1$. (c) L; $K_n(t)=O(1)$ uniformly in t. (d) Fourier-Stieltjes series; $K_n(t)$ converges uniformly.

D. Waterman (Milwaukee, Wis.)

9789:

Goes, G. Multiplikatoren für starke Konvergenz von Fourierreihen. I, II. Studia Math. **17** (1958), 299–308, 309–311.

Soit $f\in E$ et $(*)\ \sum(a_\nu\cos\nu x+b_\nu\sin\nu x)=\sum A_\nu(x)$ la série de Fourier de f; par $T\in(E,E_1)$ désignons la transformation $\sum\lambda_\nu A_\nu(x)$ lorsque la dernière série appartient à E_1. En prennant pour E un des espaces L_p $(1\le p\le\infty)$, L_ϕ (Orlicz-Zaanen) ou C, désignons par E^* leurs espaces duales, par $E_N\in E$ tels $f\in E$ pour lesquels $\lim\|s_n-f\|_E=0$, $n\to\infty$; soit $s_n(f)$ la somme partielle de $(*)$, et $K_n(t)=\sum_1^n\lambda_k\cos kt$; alors dans I l'auteur démontre: (1) soit E du type L_p $(1\le p<\infty)$, $L_\phi{}^\Delta$ (c'est-à-dire, $\phi(2u)\le M\phi(u)$), ou bien C, alors $T\in(E,C_N)$ si et seulement si $\|K_n(t)\|_{E^*}=O(1)$, $n\to\infty$. (2) Soit E_1 l'espace L_p $(1\le p\le\infty)$, L_ϕ ou C, alors $T\in(L,E_{1N})$ si et seulement si $\|K_n(t)\|_{E_1}=O(1)$, $n\to\infty$. Le cas $E=C$ dans (1) est un résultat de J. Karamata [J. Math. Pures Appl. (9) **35** (1956), 87–95; MR **17**, 964].

Dans II l'auteur montre que la même méthode d'analyse fonctionnelle utilisée dans I donne aussi, sous une forme plus générale, des résultats déjà trouvés par M. Katayama [#9788].

M. Tomić (Belgrade)

9790:

Konyuškov, A. A. The transformation of trigonometric series by means of monotone, convex, and concave sequences of multipliers. Mat. Sb. (N.S.) **51 (93)** (1960), 27–72. (Russian)

À toute série (1) $\sum_1^\infty(a_n\cos nx+b_n\sin nx)$ et à toute suite $\{\lambda_n\}$ positive on associe la série (2) $\sum_1^\infty\lambda_n(a_n\cos nx+b_n\sin nx)$. On désigne par Λ toute suite $\{\lambda_n\}$ concave tendant vers ∞, et par λ toute suite $\{\lambda_n\}$ convexe tendant vers zéro. Si A est une classe de fonctions, notons $(1)\in A$ le fait qu'existe une $f\in A$ dont la série de Fourier est (1). La plupart des résultats de l'article sont de l'un des types suivants:

(α) (th. 1, 7, 11)	$(1)\in A\Leftrightarrow\exists\Lambda$	telle que $(2)\in B$;
(α') (th. 4, 9)	$(1)\in A'\Leftrightarrow\exists\Lambda$	telle que $(2)\in B$;
(β) (th. 2, 5, 6, 8, 10)	$(1)\in B\Leftrightarrow\forall\lambda$	on a $(2)\in B$;
(β') (th. 12)	$(1)\in B\Leftrightarrow\forall\lambda$	on a $(2)\in B'$.

Dans les th. 4 et 5, A et A' [resp. B et B'] sont les classes de fonctions f vérifiant une condition $\omega_{f,p}(\delta)=o(\varphi(\delta))$ [resp. $O(\varphi(\delta))$] où $\varphi(\delta)$ est une fonction donnée, et $\omega_{f,p}(\delta)=\sup_{0\le h\le\delta}\|f(x+h)-f(x)\|_p$ le module de continuité de f dans L^p $(p\ge 1)$. Dans les autres énoncés, les conditions qui définissent ces classes sont de la forme $\|f-t_n\|_p=o(\varphi_n)$ [resp. $O(\varphi_n)$] où $p\ge 1$ et $\{\varphi_n\}$ est une suite donnée; t_n représente, suivant les cas, le polynome trigonométrique de degré $\le n$ de meilleure approximation dans L^p (th. 1, 2, 12 en ce qui concerne la définition de B'), ou la somme de Fourier d'ordre n (th. 6, 7, 12 en ce qui concerne la définition de B), ou la somme de Féjer d'ordre n (th. 8, 9), ou une somme de la Vallée-Poussin de degré n (th. 10, 11). Comme corollaire du théorème 12, il existe des fonctions $f\sim\sum_1^\infty\lambda_n n^{-\alpha}\cos nx$ [resp. $\sin nx$] avec $0<\alpha<1$ et $\lim\lambda_n=0$, qui ne satisfont pas $\omega_{f,1}(\delta)=O(\delta^\alpha)$ si $p>1$.

J.-P. Kahane (Montpellier)

9791:

Noble, M. E. Some more convergence criteria for Fourier series. Proc. London Math. Soc. (3) **10** (1960), 141–160.

Soit $f(x)$ une fonction paire, $f\in L(-\pi,\pi)$, $f(0)=0$. Le résultat essentiel de l'auteur est le suivant: Supposons que $\int_0^t|f(u)|du=o(t)$ $(t\to 0)$, et supposons qu'il existe une suite

d'entiers positifs $n_k \to \infty$ et une suite de nombres positifs $\lambda_k = o(n_k)$ $(k \to \infty)$ tels que:

$$\int_{n_k - \lambda_k^{-1}}^{n_k - \lambda_k^{-1}} u^{-1} |f(u + \pi/n_k) - f(u)|\, du = o(1) \quad (k \to \infty),$$

et tels que (pour $p > q$ et $|p, q - n_k| \leq \lambda_k$) $\liminf [\min (S_p^{(0)} - S_q(0))] \geq 0$ ($S_n(x)$ la somme de Fourier partielle de f); on a alors $S_{n_k}(0) \to 0$. *S. Mandelbrojt* (Paris)

9792a:

Efimov, A. V. Approximation of periodic functions by de La Vallée-Poussin sums. Izv. Akad. Nauk SSSR. Ser. Mat. **23** (1959), 737–770. (Russian)

9792b:

Efimov, A. V. Approximation of continuous periodic functions by Fourier sums. Izv. Akad. Nauk SSSR. Ser. Mat. **24** (1960), 243–296. (Russian)

9792c:

Efimov, A. V. Approximation of periodic functions by de La Vallée Poussin sums. II. Izv. Akad. Nauk SSSR. Ser. Mat. **24** (1960), 431–468. (Russian)

Ces trois articles font suite à une étude de l'auteur sur l'approximation des fonctions périodiques par leurs sommes de Fourier ou de Féjer [même Izv. **22** (1958), 81–116; MR **20** #3417]. Les notations varient légèrement de l'un à l'autre; nous exprimerons ici les résultats à l'aide de notations simplifiées. Pour toute fonction f continue 2π-périodique, soit

$$\omega_1(\delta, f) = \sup_{|x| \leq \pi, |h| \leq \delta} |f(x+h) - f(x)|,$$

$$\omega_2(\delta, f) = \sup_{|x| \leq \pi, |h| \leq \delta} |f(x+h) - 2f(x) + f(x-h)|.$$

L'auteur rappelle ou établit ((c), introduction) des conditions explicites pour que des fonctions $\omega_1(\delta)$ et $\omega_2(\delta)$ puissent s'écrire $\omega_1(\delta, \varphi)$ ou $\omega_2(\delta, \varphi)$; dans la suite, ces conditions sont supposées réalisées, ainsi que l'hypothèse $(*) \int_0^\delta \omega_2(t) t^{-1} dt = O(\omega_2(\delta))$ $(\delta \to 0)$ (qui n'intervient que dans certains énoncés). On désigne par H_1^ω [resp. H_2^ω] l'ensemble des fonctions f telles que $\omega_1(\delta, f) \leq \omega_1(\delta)$ [resp. $\omega_2(\delta, f) \leq \omega_2(\delta)$]. Lorsque

$$f(x) = \sum_{k=1}^{\infty} (\pi k^r)^{-1} \int_{-\pi}^{\pi} \varphi(x+t) \cos(kt + \beta\pi/2)\, dt$$

avec $r \geq 0$, $\int_{-\pi}^{\pi} \varphi(t) dt = 0$ et $\varphi \in H_1^\omega$, on note $f \in W_\beta^r H_1^\omega$; lorsque r est entier et $\beta = r$ [resp. $r+1$], cela signifie que $f^{(r)}$ [resp. la fonction conjuguée de $f^{(r)}$] appartient à H_1^ω. On désigne par S_n la n-ième somme partielle de la série de Fourier de f, et par $\sigma_{n,p}$ la somme de La Vallée-Poussin $(p+1)^{-1} \sum_{k=n-p}^{n} S_k$. En notant $\|f\| = \sup_x |f(x)|$, l'objet essentiel des trois articles est une estimation précise de $\mathscr{E}_n(C) = \sup_{f \in C} \|s_n - f\|$ [resp. $\mathscr{E}_{n,p}(C) = \sup_{f \in C} \|\sigma_{n,p} - f\|$], quand C est une des classes de fonctions définies ci-dessus. L'un des résultats les plus simples est le suivant ((a), théorème 6; (c), théorème B): si $\omega(\delta) = \delta$, et si $p \geq n/2$, on a

$$\mathscr{E}_{n,p}(H_2^\omega) = \frac{1}{\pi n} \log \frac{n}{n-p} + O\left(\frac{1}{n}\right)$$

uniformément par rapport à p. Les évaluations sont différentes suivant que $p \leq n/2$ ou $p \geq n/2$. Dans le premier cas ($p \leq n/2$) quelques complications d'énoncés apparaissent pour les classes H_2^ω et $W_\beta^r H_2^\omega$; le meilleur résultat ((c), théorèmes 5 et A) est le suivant:

$$\mathscr{E}_{n,p}(W_\beta^0 H_1^\omega) = \pi^{-1} C(n, \omega) \log \frac{n}{p+1} + 2\pi^{-1} |\sin(\beta\pi/2)|\, D(n, \omega) + O(\omega_1(1/n))$$

uniformément par rapport à p, avec

$$\pi C(n, \omega) = \sup_{f \in H_1^\omega} \left| \int_{-\pi}^{\pi} f(x) \cos nx\, dx \right| = 2 \int_0^{\pi/2} \omega_1(2t/n) \sin t\, dt,$$

$$2D(n, \omega) = \int_0^{1/n} \omega_1(2t) t^{-1}\, dt.$$

On en déduit ((c), théorème 10) une évaluation à $O(n^{-r} \omega_1(1/n))$ près de $\mathscr{E}_{n,p}(W_\beta^r H_1^\omega)$ (dans la formule 6.2, un exposant r est à rétablir). Dans le cas $p > n/2$, on a également une évaluation à $O(\omega_1(1/n))$ près de $\mathscr{E}_{n,p}(W_\beta^0 H_1^\omega)$ ((c), théorèmes 7 et A), mais l'auteur n'en déduit pas dévaluation de $\mathscr{E}_{n,p}(W_\beta^r H_1^\omega)$ pour $r > 0$; pour $\mathscr{E}_{n,p}(W_\beta^0 H_2^\omega)$ dans les cas $\beta = 0$ ou 1, on a une estimation à $O(\omega_2(1/n))$ près ((c), théorèmes 8 et B) dont un cas particulier simple a été donné ci-dessus.

Les principaux résultats de (a) et (b) se trouvent généralisés en (c). Cependant ces deux articles (surtout le second) renferment la plupart des outils utilisés. Parmi les résultats secondaires, signalons celui-ci, qui ne nécessite pas l'hypothèse (*) ((b), théorème 1): il existe une constante C telle que, si $f(x) = ax + b + g(x)$ avec $g \in H_2^\omega$, $f(0) = f(d) = 0$, on a, pour tout $x \in [0, d]$, $|f(x)| < Cx \int_x^{2d} \omega_2(t) t^{-2} dt$. {Le réferent souhaite que dans les tirés à part de l'article (b) les pages 274 et 291 soient rétablies à leur place.} *J.-P. Kahane* (Montpellier)

9793:

Ul'yanov, P. L. Divergent Fourier series of class L^p $(p \geq 2)$. Dokl. Akad. Nauk SSSR **137** (1961), 786–789 (Russian); translated as Soviet Math. Dokl. **2**, 350–354.

A brief history and summary is given of results that have been obtained concerning the divergence of trigonometric and general orthogonal Fourier series of classes L^2 and L^p $(p > 1)$, the main consideration being of series in which the natural order of the terms has been altered.

A theorem is stated to the effect that for any T^* method of summation, there is an L^p $(p > 0)$ function having a Fourier series in the Walsh normal orthogonal sequence that can be rearranged so that it is T^* non-summable almost everywhere. A similar theorem is stated for L^2-functions with the Haar normal orthogonal sequence. *F. Goodspeed* (Quebec)

9794:

Manaresi, Fabio. Su alcune serie analoghe alle serie di Fourier trigonometriche. Boll. Un. Mat. Ital. (3) **14** (1959), 360–372.

Soit $\gamma(z)$ une fonction absolument continue, croissante impaire dans $[-1, 1]$, avec $\gamma'(0)$ existant, $\gamma'(0) = 1$, $(\gamma'(z) - 1)/z$ intégrable L au voisinage de l'origine; et soit

$z=g(\tau)$ la fonction inverse de γ. L'auteur étudie la sommabilité des séries de la forme $\sum p_n(x)$, $\sum q_n(x)$,

$$\begin{matrix} p_n \\ q_n \end{matrix}(x) = \frac{1}{\rho}\int_0^\pi [f(x+t)\pm f(x-t)] \begin{matrix} \cos \\ \sin \end{matrix}\left[n\pi g\left(\frac{t}{\rho}\right)\right] dt,$$

où f est localement intégrable L et périodique de période 2π ($\rho=\pi/\gamma(1)$). *S. Mandelbrojt* (Paris)

9795:

Efimov, A. V. On linear methods of summation of Fourier series of periodic functions. Dokl. Akad. Nauk SSSR **131** (1960), 234–237 (Russian); translated as Soviet Math. Dokl. **1**, 218–221.

Soit $\Lambda=\{\lambda_k^{(n)}\}$ ($k=0, 1, \cdots, n+1$; $n=0, 1, \cdots$; $\lambda_0^{(n)}=1$, $\lambda_{n+1}^{(n)}=0$) une matrice triangulaire. A toute fonction sommable 2π-périodique $f(x)$ correspond le polynôme trigonométrique

$$U_n(f, x, \Lambda) = a_0/2+\sum_1^n \lambda_k^{(n)}(a_k\cos kx+b_k\sin kx),$$

où a_k, b_k sont les coefficients de Fourier de $f(x)$. L'auteur énonce d'abord l'estimation suivante:

$$(*)\quad \int_0^\pi |K_n(t)|\,dt \leqq C_1+C_2\sum_{k=0}^{n-1}\frac{(k+1)(n-k)}{n+1}|\Delta^2\lambda_k^{(n)}| + C_3\sum_{k=0}^{n}\frac{|\lambda_k^{(n)}|}{n-k+1},$$

$$\Delta^2\lambda_k^{(n)} = \lambda_k^{(n)}-2\lambda_{k+1}^{(n)}+\lambda_{k+2}^{(n)},$$

$$K_n(t) = \tfrac{1}{2}+\sum_1^n \lambda_k^{(n)}\cos kt.$$

De (*) on tire: si Λ satisfait à la condition

$$\sum_0^{(n-1)}\frac{(k+1)(n-k)}{n+1}|\Delta^2\lambda_k^{(n)}| < C,$$

alors $\lim \lambda_k^{(n)}=1$, $n\to\infty$, $k=1, 2, \cdots$, et

$$\sum_0^n |\lambda_k^{(n)}|/(n-k+1) < C,$$

sont les conditions nécessaires et suffisantes pour que $U_n(f, x, \Lambda)\to f(x)$ à tout point de Lebesgue. Ceci généralise les résultats de S. M. Nikol'skiĭ [Izv. Akad. Nauk SSSR Ser. Mat. **12** (1948), 259–2781; MR **10**, 247] et de B. Sz.-Nagy [Acta Sci. Math. Szeged **12** (1950), pars B, 204–210; MR **11**, 656].

Soit $H[\omega]$ la classe de fonctions continues dont le module de continuité est $\omega(\delta, f)$, et $W_\beta^r H[\omega]$ la classe de fonctions représentées par la série

$$f(x) = a_0/2+\sum_{k=0}^{\infty}(\pi k^r)^{-1}\int_{-\pi}^{\pi}\varphi(x+t)\cos(kt+\beta\pi/2)\,dt,$$

$$r\geqq 0,\quad \phi(x)\in H[\omega],\quad \int_{-\pi}^{\pi}\phi(x)\,dx = 0.$$

L'auteur énoncé alors une estimation asymptotique précise, d'ailleurs la meilleure possible, pour

$$\sup\|f(x)-U_n(f, x, \Lambda)\|_{C_{2\pi}},\quad f\in W_\beta^r H[\omega],$$

mais le résultat principal est assez long pour être reproduit ici. *M. Tomić* (Belgrade)

9796:

Varshney, O. P. On the absolute harmonic summability of Fourier series. Proc. Amer. Math. Soc. **11** (1960), 588–595.

The result of this paper is: If $f(x)$, of period 2π, is of bounded variation and if $|f(x+h)-f(x)|\leqq A\log^{-1-\varepsilon}(1/h)$ for some $\varepsilon>0$, A any positive constant, $0\leqq x\leqq 2\pi$, then the Fourier series of $f(x)$ is absolutely harmonic summable. By this is meant that the infinite series $\sum|t_n-t_{n-1}|$ converges, where t_n is the nth harmonic mean of the partial sums of the Fourier series. The conditions on f are not sufficient to ensure absolute convergence, as a consequence of a result of R. Salem [Duke Math. J. **10** (1943), 23–31; MR **4**, 156] to which the author refers. {Mr. R. Forbes has pointed out misprints in line 4 of formula (2.7), p. 591: $k(k+1)$ should be replaced by $(k+1)(k+2)$ and $\sin(n-k+\frac{1}{2})t$ should be $\sin(k-n+\frac{1}{2})t$.}

L. Lorch (Edmonton, Alta.)

9797:

Kanno, Kosi; Tsuchikura, Tamotsu. On the absolute summability of Fourier series. Tôhoku Math. J. (2) **11** (1959), 456–479.

Let $\sigma_n^{(\alpha)}$ and $\tau_n^{(\alpha)}$ denote the nth Cesàro means of order α (>-1) of the series $\sum u_n$ and the sequence $\{nu_n\}$, respectively. A series $\sum u_n$ was called $|C, \alpha|_k$ summable by Flett [Proc. London Math. Soc. (3) **7** (1957), 113–141; MR **19**, 266] if one of the following series converges (this implies that both converge):

$$\sum_n n^{k-1}|\sigma_n^{(\alpha)}-\sigma_{n-1}^{(\alpha)}|^k,\quad \sum_n n^{-1}|\tau_n^{(\alpha)}|^k.$$

The authors introduce the notion of $\{C, \alpha\}_{k,p}$ summability; this means that the series

$$\sum_n\{2^{-n}\sum_{2^n\leqq j<2^{n+1}}|\tau_j^{(\alpha)}|^p\}^{k/p}$$

converges. They prove various results comparing the two kinds of summability. The major part of the paper is devoted to an extension to $\{C, \alpha\}_{k,p}$ summability of results on $|C, \alpha|_k$ summability of a Fourier series and its conjugate series as obtained by Tsuchikura [same J. **5** (1954), 302–312; MR **15**, 866] and Flett [Proc. London Math. Soc. (3) **8** (1958), 258–311; MR **20** #6007].

J. Korevaar (Madison, Wis.)

9798:

Boyer, B. J. Summability of derived conjugate series. Pacific J. Math. **10** (1960), 1139–1145.

The author gives a condition for the summability of the successively derived conjugate Fourier series of a CP-integrable function. This is a counterpart of the author's condition for the successively derived Fourier series [B. J. Boyer, same J. **10** (1960), 475–485; MR **22** #3928].

G. Sunouchi (New Haven, Conn.)

9799:

Aljančić, S. Meilleure approximation et classes de saturation du procédé de Hölder dans les espaces C et L^p. Acad. Serbe Sci. Publ. Inst. Math. **12** (1958), 109–124.

The author studies the best order of approximation by the Hölder method of summability of Fourier series and the corresponding saturation classes, for the space C and the spaces L^p. The best order of approximation for the means $H^{(k)}$ is $(\log n)^{k-1}/n$ (k integer $\geqq 1$). The saturation

classes are (1) for the space C, functions whose conjugates are in Lip 1; (2) for the space L, functions whose conjugates are of bounded variation; (3) for the space L^p, $1<p$, functions whose conjugates are absolutely continuous with derivatives in L^p. *E. M. Stein* (Chicago, Ill.)

9800:

Rooney, P. G. On the representation of sequences as Fourier coefficients. Proc. Amer. Math. Soc. **11** (1960), 762–768.

The author characterizes the sequences of Fourier coefficients $\{c_n\}$ arising from functions in $L_p(0, 2\pi)$ $(1<p\leq 2)$ as follows. There exists a constant M so that $(k+1)^{-p}\sum_{m=0}^{k}|\mu_{k,m}(c)|^p \leq M$ $(k=0, 1, 2, \cdots)$, where $\mu_{k,m}(c)=\sum_{-\infty}^{+\infty} c_r a_{r,k,m}$, and

$$a_{r,k,m} = C_{k,m}\int_0^1 t^m(1-t)^{k-m}e^{\pi i r t}\,dt.$$

$C_{k,m}$ are the binomial coefficients. This characterization may be considered as the analogue of a similar one given by the author for Fourier transforms [Canad. J. Math. **11** (1959), 168–174; MR **22** #1790].

E. M. Stein (Chicago, Ill.)

9801:

Paneyah, B. P. Some theorems of Paley-Wiener type Dokl. Akad. Nauk SSSR **138** (1961), 47–50 (Russian); translated as Soviet Math. Dokl. **2**, 533–536.

For a bounded open set $\Omega\subset R^n$, let $\mathscr{H}$ be the space of all Fourier transforms

$$\hat{u}(\xi) = (2\pi)^{-n/2}\int_\Omega \exp[i\langle\xi, x\rangle]u(x)\,dx$$

of functions in $L_2(\Omega)$. Let $\mathscr{M}$ be a set of positive measure in R^n, and $\|u\|_{\mathscr{M}}=(\int_{\mathscr{M}}|u(\xi)|^2d\xi)^{1/2}$. A number of results are announced concerning conditions under which $\|\cdot\|_{\mathscr{M}}$ and $\|\cdot\|_{R^n}$ are equivalent. *E. Hewitt* (Seattle, Wash.)

9802:

Rudin, Walter. Fourier-Stieltjes transforms of measures on independent sets. Bull. Amer. Math. Soc. **66** (1960), 199–202.

There is a compact perfect independent set Q in the real line, and a positive measure σ with support contained in Q, such that the Fourier-Stieltjes transform of σ vanishes at ∞. *E. Hewitt* (Seattle, Wash.)

9803:

Komatu, Yûsaku. On the integral representation of Herglotz and the trigonometric moment problem. Japan. J. Math. **29** (1959), 35–42.

Let $\{\lambda_n\}$ be a sequence of complex constants, $\lambda_{-n}=\bar{\lambda}_n$. The problem of determining a bounded increasing function $\rho(\phi)$ satisfying the condition $\int_{-\pi}^{\pi}e^{in\phi}d\rho(\phi)=\lambda_n$, $n=0, \pm 1, \pm 2, \cdots$, is called the trigonometric moment problem. Let $\mathscr{F}$ be the class of analytic functions $\Phi(z)$ which are regular in $|z|<1$ for which the quantity

$$\rho(r,\phi) = (2\pi)^{-1}\int_{-\pi}^{\phi} R[\Phi(re^{i\theta})]\,d\theta \qquad (-\pi\leq\phi\leq\pi, 0\leq r\leq 1)$$

is bounded and of bounded variation uniformly with respect to r. The Herglotz integral representation of $\Phi(z)$ is given by

$$\Phi(z) = \int_{-\pi}^{\pi}\frac{e^{i\phi}+z}{e^{i\phi}-z}\,d\rho(\phi)+i\,\mathrm{Im}[\Phi(0)],$$

where $\rho(\phi)$ is a real-valued function of bounded variation.

The author uses the Herglotz representation to obtain necessary and sufficient conditions for the solvability of the trigonometric moment problem, as well as for the uniqueness of the solution, in terms of functions belonging to $\mathscr{F}$. As is mentioned in the paper, most of the results have been found previously by other methods.

W. C. Royster (Lexington, Ky.)

9804:

Komatu, Yûsaku. On the trigonometric moment problem with reference to a class of analytic functions in an annulus. Kôdai Math. Sem. Rep. **12** (1960), 15–20.

The results of the above paper are generalized to functions regular and single valued in the annulus $q<|z|<1$, $0\leq q<1$, by employing the Villat-Stieltjes integral representation for functions of a class $\mathscr{F}_q$ which are the analogs of the functions of the class $\mathscr{F}$ in the above paper.

W. C. Royster (Lexington, Ky.)

9805:

Hartman, S.; Ryll-Nardzewski, C. Über die Spaltung von Fourierreihen fastperiodischer Funktionen. Studia Math. **19** (1960), 287–295.

Suppose $\{\lambda_k\}$, a countable set of real numbers, is the disjoint union of sets $E_1, E_2, E_3, \cdots$ which are arithmetically independent in the following sense: If $\{c_k\}$ is a sequence of integers in which only finitely many are different from 0 and if $\sum c_k\lambda_k=0$, then $\sum^{(n)}c_k\lambda_k=0$ for $n=1, 2, 3, \cdots$; the superscript (n) here indicates that the summation is extended only over those k for which $\lambda_k\in E_n$. Theorem: If $f(t)\sim\sum a_k\exp(i\lambda_k t)$ is an almost periodic function, then (1) each of the series $\sum^{(n)}a_k\exp(i\lambda_k t)$ is the Fourier series of an almost periodic function f_n, (2) the series $\sum f_n$ converges uniformly to f, and (3) $\sum\sup_t|f_n(t)|<\infty$. The authors actually formulate and prove this result for general abelian groups in place of the real line.

W. Rudin (Madison, Wis.)

9806:

Kahane, Jean-Pierre. Quelques généralisations de la notion de périodicité. Textos de Mat., No. 6, Conf. 3, 12 pp. Instituto de Física e Matmática, Universidade do Recife, 1960.

This expository lecture begins with the observation that the obvious definition of periodicity (i.e., $f(t-a)=f(t)$ for some $a>0$ and all t) can be, and has been, modified by calling a function "periodic" if it is a limit, in some sense or other, of a sequence of trigonometric polynomials. Some properties of quasi-periodic functions (Bohl and Esclangon), almost periodic functions and distributions (Bohr, Stepanoff, Besicovitch, Schwartz), mean-periodic functions (Delsarte) and pseudo-periodic functions (Paley-Wiener) are discussed from this point of view.

W. Rudin (Madison, Wis.)

9807:

Timan, M. F. Absolute convergence of multiple Fourier series. Dokl. Akad. Nauk SSSR **137** (1961), 1074–1077 (Russian); translated as Soviet Math. Dokl. **2**, 430–433.

The author points out that imposing conditions of S. Bernstein's type on a function of several variables with respect to each variable separately, so that the Fourier series of the function converges absolutely in each variable, is not enough to make the multiple Fourier series of the function converge absolutely. He introduces partial best L_p approximations E_{n_ν} and partial L_p moduli of smoothness $\omega_{r,\nu}$ of order r (each with respect to the νth variable) and gives the following theorems. If $\alpha_\nu > 0$ and $\sum \alpha_\nu = 1$, and for each ν we have $\sum_{n_\nu=1}^{\infty} n_\nu^{-1/2} E_{n_\nu}^{\alpha_\nu}(f) < \infty$ (L_2 metric), then the Fourier series of f converges absolutely. If for each ν we have $\sum_{n_\nu=1}^{\infty} n_\nu^{k/2-1} E_{n_\nu}(f) < \infty$ then the Fourier series of f converges absolutely. These are, in a certain sense, best possible results. There are equivalent formulations in terms of the moduli of smoothness. Generalizations are given in directions that would be expected from the one-variable case.

R. P. Boas, Jr. (Evanston, Ill.)

9808:

Choquet, Gustave; Deny, Jacques. Sur l'équation de convolution $\mu = \mu * \sigma$. C. R. Acad. Sci. Paris **250** (1960), 799–801.

Consider the convolution equation (1) $\mu = \mu * \sigma$, where σ is a given positive measure on a locally compact Abelian group G. If $\sigma(G) = 1$, the bounded solutions of (1) are exactly the bounded periodic distributions whose group of periods contains the support of σ. For $0 < \sigma(G) \leqq \infty$, the solutions of (1) are also obtained; here the result is more complicated. *E. Hewitt* (Seattle, Wash.)

9809:

Simon, Arthur B. Symmetry in measure algebras. Bull. Amer. Math. Soc. **66** (1960), 399–400.

Let $\mathfrak{M}$ be the maximal ideal space of $M(R)$, the Banach algebra of all bounded complex Borel measures on the line R, with convolution as multiplication. Since R is its own dual group, $R \subset \mathfrak{M}$. Let S be the largest subset of $\mathfrak{M}$ on which the Gelfand transforms of $M(R)$ are closed under conjugation. It is clear that S contains $\bar{R}$, the closure of R in $\mathfrak{M}$, and it is known that $S \neq \mathfrak{M}$. The author proves that S is larger than $\bar{R}$. *W. Rudin* (Madison, Wis.)

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See also 9575, 9629, 9631, 9776, B10563.

9810:

Klamkin, M. S.; Newman, D. J. The philosophy and applications of transform theory. SIAM Rev. **3** (1961), 10–36.

An expository paper giving a variety of examples of transforming problems into simpler ones. Transformations here include geometric interpretations of problems in analysis, conformal mapping, and integral transformations.

R. V. Churchill (Ann Arbor, Mich.)

9811:

Widder, David V. Transformations associées à l'équation de la chaleur. C. R. Acad. Sci. Paris **251** (1960), 915–917.

This paper considers relations among the following transformations associated with the heat equation (i) $u_t = u_{xx}$:

$$(1) \qquad P_t f(x) = \int_{-\infty}^{\infty} k(x-y, t) f(y)\, dy,$$

where

$$k(x, t) = (4\pi t)^{-1/2} \exp(-x^2/4t), \qquad t > 0,$$

$$(2) \qquad F_t f(x) = (2\pi)^{-1} \int_{-\infty}^{\infty} \exp(ixy - ty^2) f(y)\, dy$$

and

$$(3) \qquad Au(x, t) = k(x, t) u(x/t, -1/t),$$

each of which gives rise to an integral of (i) in a certain functional domain.

Let $u(x, t)$ belong to the class H^*, introduced in P. C. Rosenbloom and D. V. Widder, Trans. Amer. Math. Soc. **92** (1959), 220–266 [MR **21** #5845], for $a < t < b$, that is, u is an integral of (i) having continuous derivatives of second order and satisfying

$$u(x, t) = P_{t-t'} u(x, t'), \quad a < t' < t < b.$$

The following theorems are given:

I. If $f(x) \in L(-\infty, \infty)$, then $P_t f \in H^*$ for $0 < t < \infty$ and $P_t f = F_t \phi$, where $\phi(x) = \int_{-\infty}^{\infty} \exp(-ixy) f(y) dy$.

II. If $u(x, t) \in H^*$ for $a < t < b$ and if $a < t_0 < b$, then

$$F_t u(-2ix, t_0) = Au(x, t + t_0) \quad (t > \sigma^{-1})$$

and $u(-2ix, t_0)$ is an entire function of order $\leqq 2$ and type $\leqq \sigma^{-1}$, $\sigma = \min(t_0 - a, b - t_0)$.

The proof of II uses results from the above quoted paper.

III. If $u(x, t) \in H^*$ for $a < t < b$ and if $a < t_0 < b$, then

$$u(-2x, t_0) = \int_{-\infty}^{\infty} \exp(-xy - tx^2) Au(y, t + t_0)\, dy \qquad (t > \sigma^{-1}),$$

$\sigma = \min(t_0 - a, b - t_0)$.

III is verified in a special case.

J. Mitchell (University Park, Pa.)

9812:

Narayana Rao, M. L. Some general kernels for the derivation of self-reciprocal functions. Proc. Indian Acad. Sci. Sect. A **50** (1959), 385–391.

Let T_ν be the Hankel transform by $T_\nu f(x) = \int_0^\infty f(t)(xt)^{1/2} J_\nu(xt)\, dt$. A function f is in R_ν (is self-reciprocal) if $T_\nu f = f$. The author investigates a class of integral transforms H that carry R_ν into R_μ, and from these, he obtains more examples of functions which are self-reciprocal. His examples of transforms H are of the form $Hf(x) = \int_0^\infty f(t) h(tx)\, dt$ where

$$h(x) = s^{1/2} K_{(\mu-\nu)/2}(\alpha s) J_{(\mu+\nu)/2}(\beta s)$$

and $\alpha^2 + \beta^2 = 1$. *R. C. Buck* (Madison, Wis.)

9813:

Masood, S. Some theorems on self-reciprocal functions. Proc. Indian Acad. Sci. Sect. A **53** (1961), 35–43.

Denote by R_μ the class of functions self-reciprocal with respect to the Hankel transform of order μ. It has been shown by Mohan (=Mehrotra) [Proc. London Math. Soc. (2) **34** (1932), 231–240] that if $F \in R_\mu$, and $g(x) = \int_0^\infty P(xy)F(y)dy$ where

$$P(x) = (2\pi i)^{-1} \int_{k-i\infty}^{k+i\infty} 2^s \Gamma(\tfrac{1}{4}+\tfrac{1}{2}\mu+\tfrac{1}{2}s) \times \Gamma(\tfrac{1}{4}+\tfrac{1}{2}\nu+\tfrac{1}{2}s)w(s)x^{-s}\,ds,$$

k and w satisfying $0<k<1$ and $w(s)=w(1-s)$, then $g \in R_\nu$.

The author considers several converse problems. For example, he shows that if $F \in R_\mu$ and $g \in R_\nu$, then P is given by the above formula, and that if $g \in R_\nu$ and P is as above, then $F \in R_\mu$. He also considers similar problems arising when F and g are connected by other integral relations, thus finding converses for various other theorems of Mohan. Proofs are largely formal.

P. G. Rooney (Toronto)

9814:

Nixon, Floyd E. ★Handbook of Laplace transformation: Tables and examples. Prentice-Hall Electrical Engineering Series. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1960. ix+115 pp. $4.95.

A useful collection of Laplace transform pairs, with enough background information on theory for anyone without previous knowledge of Laplace transforms to be able to use the book. There are also sections on determinants, on root-finding methods and Routh's criterion and on commonly used identities.

A clearly written handbook for engineers, applied mathematicians, and physicists, with a surprising amount of information for its size.

W. Freiberger (Providence, R.I.)

9815:

Isaacs, G. L. On the summability-abscissae of Laplace integrals. Proc. London Math. Soc. (3) **10** (1960), 461–479.

This note concerns the formal Laplace-Stieltjes integral

$$\int_0^\infty e^{-us}\,dA(u) \tag{1}$$

where $s=\sigma+jt$ (σ, t real), and $A(u)$ denotes a function of bounded variation which is 0 at $u=0$. For a fixed $k \geqq 0$, define σ_k to be the lower bound of numbers σ for which (1) is summable (C, k). It can be shown that when $\sigma_k > 0$,

$$\sigma_k = \limsup_{\omega\to\infty} \omega^{-1} \log |A_k(\omega)|, \tag{2}$$

and when $\sigma_k < 0$ and k is a non-negative integer,

$$\sigma_k = \limsup_{\omega\to\infty} \omega^{-1} \log |R_k(\omega)|, \tag{3}$$

where, for any $\nu > -1$, $\omega \geqq 0$,

$$R_\nu(\omega) = R_\nu A(\omega) = \underset{(C,k)}{R_\nu A(\omega)} = [\Gamma(\nu+1)]^{-1} \int_{\underset{(C,k)}{0}}^{\infty} (x-\omega)^\nu\,dA(x). \tag{4}$$

Equation (2) may be proved by applying M. Riesz's "limitation theorem" and a related one. Equation (3) was shown by Kuniyeda who also conjectured that (3) might hold for fractional k. In this note, the author finds that the answer is not completely in the affirmative. He defines $\bar{k}$ = lower bound of numbers $k \geqq 0$ for which $\sigma_k < \infty$ and gives the main result as theorem T: If $k > \bar{k}$ and $\sigma_k < 0$, then σ_k is given by (3). If $\bar{k}$ is fractional and $\sigma_{\bar{k}} < 0$, then (3) need not hold at $k = \bar{k}$.

The positive part of theorem T is a consequence of the author's following theorems. Theorem A′: If k is positive and fractional and $k > \bar{k}$, and if (1) is summable (C, k) for some s such that $\sigma < 0$, then as $\omega \to \infty$, $R_k(\omega) = o\{e^{\omega(\sigma+\varepsilon)}\}$ for every $\varepsilon > 0$. Theorem B′: If k is positive and fractional, $k > \bar{k}$ and σ_k is negative, and if $R_k(\omega) = O(e^{\omega\sigma'})$ as $\omega \to \infty$, where $\sigma' < 0$, then (1) is summable (C, k) whenever $\sigma > \sigma'$.

Y. H. Ku (Philadelphia, Pa.)

9816a:

Talbot, A. The evaluation of integrals of products of linear system responses. I. Quart. J. Mech. Appl. Math. **12** (1959), 488–503.

9816b:

Talbot, A. The evaluation of integrals of products of linear system responses. II. Continued-fraction methods. Quart. J. Mech. Appl. Math. **12** (1959), 504–520.

The author gives methods for the evaluation of the integrals $\int_0^\infty t^r xy\,dt$ and $\int t^r x^2\,dt$ ($r=0, 1, 2, \cdots$) where x and y are functions of t which have rational Laplace transforms. In I, a Parseval formula is developed; $\int_0^1 xy\,dt = (2\pi i)^{-1} \int_C \bar{x}(p)\bar{y}(-p)dp$, where $\bar{x} = h/f$ and $\bar{y} = k/g$ are the rational transforms of x and y, resp., and are such that the integrand is of order $O(p^{-2})$ and the poles of $\bar{x}$ are to the left of those of $\bar{y}$. This allows the use of the Cauchy residue theorem to get $\int xy\,dt = \sum_i (hk/f'g)_{p=p_i}$, where p_i are the zeros of f. In order to evaluate the sum, a specialized partial fraction technique is used in which it is necessary to solve an equation of the form $fv+gu=hk$ for u and v. This is done by replacing this equation by a set of simultaneous equations in the coefficients of u and v. This same idea is then extended to permit the evaluation of the other integrals indicated above.

In II, a method is developed for dealing with the equation $fv+gu=hk$ which is based on the development of g/f in continued-fraction form and is valid as long as the zeros of f are simple. It is claimed that his method is more economical than standard methods.

Both papers give examples illustrating the techniques.

C. G. Maple (Ames, Iowa)

9817:

Jury, E. I.; Pai, M. A. On the summation of $\sum_{n=0}^\infty f_n/n^k$ and its associated integrals. J. Franklin Inst. **271** (1961), 79–93.

Convolutions of z-transforms are evaluated by means of Laplace transforms, and formal results are obtained which are then used to deduce well-known results, such as $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots=\frac{1}{4}\pi$, and the divergence of $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$.

A. P. Guinand (Saskatoon, Sask.)

9818:

Bhonsle, B. R. Some recurrence relations and series for the generalised Laplace transform. Proc. Glasgow Math. Assoc. **4**, 119–121 (1960).

The author considers the following generalisation of Varma's generalisation of the Laplace transform:

$$\phi_{k,m,\lambda}(p) = p\int_0^\infty e^{-px/2}(px)^{m-1/2}W_{k,m}(px)x^\lambda h(x)\,dx \qquad (\mathrm{Re}\,p > 0).$$

The main results of the paper are the following recurrence formulae:

$$p\phi_{k,m-1,\lambda+1}(p)+(\tfrac{1}{2}-k+m)\phi_{k-1,m,\lambda}(p) = p(\tfrac{1}{2}-k-m)\phi_{k-1,m-1,\lambda+1}(p)+\phi_{k,m,\lambda}(p);$$

$$(m+k-\tfrac{1}{2})\phi_{k,m,\lambda}(p)-p\phi_{k,m,\lambda+1}(p) = \{m^2-(k-\tfrac{1}{2})^{1/2}\}\phi_{k-1,m,\lambda}(p)-p\phi_{k+1/2,m-1/2,\lambda+1}(p).$$

In the end the author derives some recurrence formulae and series for MacRobert's E-function.

B. Mohan (Banaras)

9819:

Wąsowski, J. Zwei Lehrsätze in der Operatorenrechnung. Z. Angew. Math. Mech. **41** (1961), 9–17. (English and Russian summaries)

Formulierung und Beweis von Präzisierungen bekannter Rechnungs-Regeln der Laplacetransformation, die bei ihrer rechnerischen Anwendung z.B. bei der Auflösung von Differentialgleichungssystemen nützliche Dienste leisten.

W. Saxer (Zürich)

9820:

Lavoine, Jean. Transformées de Fourier inverses de $e^{-i2\pi t_0\sqrt{(q^2+M^2)}}$ et de $e^{-i2\pi t_0\sqrt{(q^2+M^2)}}/2\sqrt{(q^2+M)}$, $q^2=p_1^2+p_2^2+p_3^2$, fonctions singulières de l'électrodynamique. C. R. Acad. Sci. Paris **250** (1960), 2318–2320.

The author finds the three-dimensional Fourier transform of $g(t_0, q)=\exp(-2\pi i t_0(q^2+M^2)^{1/2})$, $q^2=p_1^2+p_2^2+p_3^2$, and of $h(t_0, q)=(q^2+M^2)^{-1/2}g(t_0, q)$ as distributions in the sense of L. Schwartz. Typical result: The Fourier transform of h is

$$PVY(u)Mu^{-1/2}K_1(2\pi Mu^{1/2})+\tfrac{1}{2}iPVY(-u) \times\pi M(-u)^{-1/2}H_1^{(2)}(2\pi M(-u)^{-1/2})-i(4t_0)^{-1}\delta(y-t_0),$$

where PV means principal value, $Y(u)=\frac{1}{2}(1+\mathrm{sgn}\,u)$, δ is the Dirac distribution, $y^2=x_1^2+x_2^2+x_3^2$, and $u=y^2-t_0^2$.

H. O. Pollak (Murray Hill, N.J.)

9821:

Meller, N. A. Operational calculus for the operator $B_\alpha=t^{-\alpha}d/dt\,t^{\alpha+1}d/dt$. Vyčisl. Mat. **6** (1960), 161–168. (Russian)

The operational calculus for B_α is constructed, generalizing the results of Ditkin [Dokl. Akad. Nauk SSSR **116** (1957), 15–17; MR **22** #882] for the case $\alpha=0$. The set of functions twice differentiable on the positive axis and with $t^\alpha f(t)$ summable over every finite interval is shown to be a ring with convolution defined by

$$f(t)*g(t) = \frac{1}{\Gamma(1+\alpha)\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t(t-u)^{-\alpha}du \times\frac{d}{du}\,u\frac{d}{du}\int_0^u dv\int_0^1 dx\,v^\alpha(1-x)^\alpha f(xv)g[(1-x)(u-v)],$$

and it is proved that $B_\alpha^{-1}f(t)=D^{-1}t^{-\alpha-1}D^{-1}t^\alpha f(t)$ in this ring, with $D^{-1}f(t)=\int_0^t f(u)du$. Also, $B_\alpha^{-\tau}*B_\alpha^{-\mu}=B_\alpha^{-(\tau+\mu)}$. ($F(B_\alpha)$ stands for the function $F(B_\alpha)1$.) The forms of various specific functions of B_α are given; the relation to standard operational calculus is provided by the formulae: If $F(B_\alpha)=f_1(t)$ and $F(D)=f(t)$, then $f(t)=[\Gamma(\alpha+1)]^{-1}\times\int_0^\infty u^\alpha f_1(tu)e^{-u}du$.

J. L. B. Cooper (Cardiff)

9822:

Weston, J. D. Positive perfect operators. Proc. London Math. Soc. (3) **10** (1960), 545–565.

The class H of all complex-valued continuous functions $x(t)$, $t\geqq 0$, is a commutative ring (when $(x+y)(t)=x(t)+y(t)$, $(x*y)(t)=\int_0^t x(u)y(t-u)du$), which has no divisors of zero. Any subring $H_1\subset H$ can be isomorphically imbedded in a field of quotients. If H_1 contains $h(t)\equiv 1$, then every field containing H_1 will contain h^{-1}. Since $(h*x)(t)=\int_0^t x(u)du$, $(h^{-1}y)(t)$ is a certain generalized "derivative" of $y(t)$. Mikusinski was the first to use these ideas for a foundation of the Heaviside calculus, taking $H_1\equiv H$. The author considers H_1 strictly contained in H, consisting of $h(t)\in H$ such that $|h(t)|=O(e^{st})$, $t\to\infty$ and minimal enlargements of H_1 (minimal rings containing H_1 and the "derivatives" of $h(t)\in H_1$ of all orders). In this manner he obtains a more particular theory, in which many results are analogous to some theorems on distributions of Schwartz. He defines the set $\mathscr{D}_0$ of all perfect functions ($h(t)\in H_1$ such that all derivatives $h^{(n)}(t)$ exist and are in H_1, and $h^{(n)}(0)=0$, $n=0, 1, \cdots$). Next are defined the perfect operators, which are mappings A of $\mathscr{D}_0$ into itself such that $A(p*q)=(Ap)*q$, $p, q\in\mathscr{D}_0$ (as, for example, the derivation operator $d/dt=D$). These operators are found to be linear (§ 4 of the present paper). As was already proved by the author in other papers, for every perfect A there is an $n\geqq 0$ and an $x(t)\in H_1$, such that $(Ap)(t)=\int_0^t x(t-u)p^{(n)}(u)du$, for every $p(t)\in\mathscr{D}_0$ and $t\geqq 0$, and the set $\mathscr{A}$ of the perfect operators A is a minimal enlargement of H_1.

In the present paper are considered positive perfect operators (such that $(Ap)(t)\geqq 0$ for every perfect $p(t)\geqq 0$). A principal result (theorem 1) is the fact that every perfect operator is representable as

$$(Ap)(t) = \int_0^t p(t-u)\,d\mu(u), \quad t\geqq 0,$$

where $\mu(t)$ is a non-decreasing function in H_1. It follows (theorem 2) that a perfect operator X which is defined by a Lebesgue integrable function $x(t)$ in the formula $(Xp)(t)=\int_0^t x(t-u)p(u)du$, $t\geqq 0$, is positive if and only if $x(t)\geqq 0$, a.e. These representation theorems for perfect operators are analogous to the local representation of distributions as "derivatives" of continuous functions and with the representation of positive distributions as measures.

Other topics considered in this paper are the Laplace transform of positive perfect operators and the uniform convergence for sequences of perfect operators. There are some indications (insufficient in reviewer's opinion) on possible—physical—applications of this theory.

S. Zaidman (Milan)

9823:

Cossar, J. Hilbert transforms and almost periodic functions. Proc. Cambridge Philos. Soc. **56** (1960), 354–366.

Part I is concerned with the existence of the Hilbert transform

$$Hf = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(t)}{t-x}\,dt$$

(where P denotes the Cauchy principal value) and the validity of the inversion formula $H^2f = -f$. Part II gives an application to B^p almost periodic (a.p.) functions.

Notation: $\overline{M}(f) = \limsup_{T\to\infty} (\int_{-T}^{T} f)/2T$; when the corresponding limit exists it is denoted by $M(f)$. $\|f\| = \{M(|f|^p)\}^{1/p}$ (when this exists).

Theorem 1: Suppose (a) f is locally L^p $(p>1)$, i.e., $|f|^p$ is integrable over every finite interval, (b) $\int_1^\infty [f(t)-f(-t)]/t\,dt$ exists and (c) $\int_T^\infty [f(t)-f(-t)]/t^2\,dt = o(1/T)$ $(T\to\infty)$. Then (i) $g = Hf$ exists and is locally L^p, (ii) $f = -Hg$, where the integral defining Hg exists in the $(C, 1)$ sense at $\pm\infty$ and (iii) $\overline{M}(|g|^p) \le A_p{}^p \overline{M}(|f|^p)$, where A_p is a constant depending on p only.

Theorem 2: Suppose that f is B^p a.p., $M(f) = 0$, $\|f\| \neq 0$ and $f \sim \sum (a_n \cos \lambda_n x + b_n \sin \lambda_n x)$ $(\lambda_n > 0)$. Then the conjugate series $\sum (b_n \cos \lambda_n x - a_n \sin \lambda_n x)$ is the Fourier series of a B^p a.p. function g such that $\|g\| \le A_p \|f\|$. Also f and g are equivalent, in the sense of the B^p distance, to B^p a.p. functions f^*, g^* such that, almost everywhere, $g^* = Hf^*$ and $f^* = -Hg^*$. *H. Burkill* (Sheffield)

INTEGRAL AND INTEGRODIFFERENTIAL EQUATIONS

See also 9638, 9715, 9855, 9857, 9861.

9824:

Friedrich, Joachim. Bemerkung zur Abelschen Integralgleichung. Z. Angew. Math. Phys. **11** (1960), 191–197. (English summary)

Eine Reihe von physikalischen Problemen, von denen Verf. einige andeutet, führen auf Integralgleichungen vom Abelschen Typus: $f(s) = \int_0^s (s^p - t^p)^\beta y(t)\,dt$. Diese für $p=1$ und $-1<\beta<0$ wohlbekannte Gleichung wird allgemein für $p\neq 0$ und $-1<\beta<n$ gelöst. Sie geht übrigens durch eine Variablentransformation auf den von R. Rothe [Math. Z. **33** (1931), 375–387] behandelten Fall $p=1$, $-1<\beta<n$ über und lässt sich als Integralgleichung vom Faltungstypus in einfacher Weise auch mit Hilfe der Laplacetransformation behandeln. *A. Pfluger* (Zürich)

9825:

Sneddon, Ian N. The elementary solution of dual integral equations. Proc. Glasgow Math. Assoc. **4**, 108–110 (1960).

The pair of dual integral equations

$$\int_0^\infty \xi^{-1}\psi(\xi) J_0(\xi\rho)\,d\xi = f(\rho) \quad (0<\rho<1),$$

$$\int_0^\infty \psi(\xi) J_0(\xi\rho)\,d\xi = 0 \quad (\rho>1)$$

is shown to be readily solved by the substitution $\psi(\xi) = \xi \int_0^1 \phi(t)\cos(\xi t)\,dt$. Similarly the pair

$$\int_0^\infty \xi\psi(\xi) J_0(\xi\rho)\,d\xi = f(\rho) \quad (0<\rho<1),$$

$$\int_0^\infty \psi(\xi) J_0(\xi\rho)\,d\xi = 0 \quad (\rho>1)$$

is solved by the substitution $\psi(\xi) = \int_0^1 \chi(t)\cos(\xi t)\,dt$ with $\chi(0) = 0$. *E. T. Copson* (St. Andrews)

9826:

Picone, Mauro. Sull'equazione integrale non lineare di seconda specie di Fredholm. Math. Z. **74** (1960), 119–128.

The author considers the non-linear Fredholm equation $u(x) = f(x) + \int_A K[x, y, u(y)]dy$, where x and y are points in a given set A, u is a point in a prescribed set B, and $K[x, y, u]$ has the form $H(x, y, u)u$, where H denotes a square matrix defined on the Cartesian product set $A\times A\times B$. If H satisfies a certain Lipschitz condition, it is shown that there exists a unique solution of the integral equation if the measure of A is sufficiently small. This is then applied to the Volterra case which the author has already considered [Ann. Mat. Pura Appl. (4) **49** (1960), 1–10; MR **22** #2867]. *I. A. Barnett* (Cincinnati, Ohio)

9827:

Smith-White, W. B. Continuously infinite matrices and the algebraic theory of Fredholm's equation. J. Austral. Math. Soc. **1** (1959/61), 134–142.

This paper gives a new account of the determinant-free theory of linear integral equations of the second kind. The author uses matrix notation, regarding the kernel $K(x, y)$ as a matrix of type $\infty\times\infty$, and a set of functions $(p_1(x), \cdots, p_m(x))$ as a matrix of type $\infty\times m$; integration replaces summation in appropriate matrix products. The (associative) dot product of two matrices K and L is defined by $K\cdot L = K + L - KL$. An $\infty\times\infty$ matrix K is said to be regular if there exists K_1 with $K\cdot K_1 = K_1\cdot K = 0$; under the dot product the regular matrices form a group with neutral element 0. Two matrices K and K' are equivalent if there exist regular matrices L and M with $K' = L\cdot K\cdot M$.

On this basis the author gives a very neat account of the determinant-free theory. The essential lemma is that if K is an $\infty\times\infty$ matrix, defined by a continuous or L^2 function, then there exists an equivalent matrix of finite rank. *F. Smithies* (Cambridge, England)

9828:

Picone, Mauro. Ancora sullo spettro in un parametro da cui dipendono certe equazioni integrali lineari. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **28** (1960), 743–745.

The note corrects an error pointed out in the abstract of an earlier note [same Atti (8) **23** (1957), 347–354; MR **20** #7195] and proves that for λ in the spectrum of the equations (1) of the abstract, the stronger condition $|\lambda|>1$ holds, provided, as in the previous note, that the kernels $H_{ij}(x, y)$ satisfy a positive definite condition. *T. H. Hildebrandt* (Ann Arbor, Mich.)

9829:

Pogorzelski, W. On the properties of a singular integral in space and their application to a system of singular integral equations. Problems of continuum mechanics (Muskhelishvili anniversary volume), pp. 327–341. SIAM, Philadelphia, Pa., 1961.

Let $N(x)$ be a function in E_n which can be written $N(x) = |x|^{-n}K(x/|x|)$, where K satisfies the condition $\int_\omega K(x')dx' = 0$, ω being the surface of the unit sphere. If

Ω is a region in E_n the author considers singular integrals of the type,

$$\Phi(x, t) = \int_\Omega N(x-y)g(y, t)\,dy.$$

With certain restrictions on the function g the integral is shown to satisfy a kind of generalized Hölder condition. The information obtained is used to apply the Schauder theory to the system of singular integral equations,

$$\varphi_\nu(x) = f_\nu(x) + \int_\Omega N_\nu(x-y)F_\nu[x, y, \varphi_1(y), \cdots, \varphi_m(y)]\,dy.$$

For suitable regularity conditions on the F_ν and f_ν the existence of at least one solution is established.

R. C. MacCamy (Pittsburgh, Pa.)

9830:

Widom, Harold. Singular integral equations in L_p. Trans. Amer. Math. Soc. **97** (1960), 131–160.

An L_p theory of the singular integral equation $\mathbf{K}f = g(x)$ is given, where $\mathbf{K}$ is the operator $\mathbf{K}_0 f + \mathbf{M}f$,

$$\mathbf{K}_0 f = A(x)f(x) + B(x)\mathbf{T}f, \quad \mathbf{T}f = \frac{1}{\pi i}\int_E \frac{f(y)\,dy}{y-x},$$

E is a set of positive measure in $(-\infty, \infty)$, $A(x)$ and $B(x)$ are essentially bounded on E, $\mathbf{M}$ is a completely continuous operator on $L_p(E)$, $g(x) \in L_p(E)$ $(p>1)$ is given and $f(x) \in L_p(E)$ is to be found. Since, according to the standard method, the solution of a singular integral equation like this is based on the representation of $f(x)$ in the form $f = f_+ + f_-$, where f_+, f_- are the limit functions of a function analytic in the upper or lower half plane, respectively, and as $2f_+ = f - i\mathfrak{H}f$, $2f_- = f + i\mathfrak{H}f$, the author first investigates Hilbert's operator $\mathfrak{H}f$. He finds weight functions $\rho(x)$ such that

$$\int_F |\rho(x)\mathfrak{H}f|^p\,dx \leqq A^p \int_E |\rho(x)f(x)|^p\,dx$$

$$(F \subset (-\infty, \infty);\ \text{meas } F > 0);$$

this inequality, a generalisation of the classical Riesz result, plays an important part in the sequel. Then the non-homogeneous case of the Riemann-Hilbert problem $G(x)f_+(x) = f_-(x) + g(x)$ is discussed. The equation $\mathbf{K}_0 f = g(x)$ is reduced to it by the standard transformation $g_1(x) = g(x)/\{A(x) - B(x)\}$; $G(x) = \{A(x) + B(x)\}/\{A(x) - B(x)\}$ for $x \in E$, $= 1$ for $x \notin E$, and in Theorem 4 the spectrum of the operator Tf (i.e., the case $A(x) = -\Lambda$, $B(x) = 1$ of $\mathbf{K}_0 f = g$) is dealt with. Finally the general equation $\mathbf{K}f = g$ is reduced to $\mathbf{K}_0 f = g$. *H. Kober* (Birmingham)

9831:

Halanay, Aristide. Solutions périodiques des systèmes généraux à retardement dans le cas de la résonance. C. R. Acad. Sci. Paris **251** (1960), 1856–1858.

Necessary and sufficient conditions are given for the existence of periodic solutions to an integro-differential equation system in terms of periodic solutions to the adjoint integro-differential equation system. The result is specialized to difference-differential equations.

E. Pinney (Berkeley, Calif.)

FUNCTIONAL ANALYSIS

See also 9565, 9566a–b, 9575, 9748, 9820, 9822, 9962, B10310.

9832:

Krishnamurthy, V. On the state of a linear operator and its adjoint. Math. Ann. **141** (1960), 153–160.

A continuous linear transformation T from a locally convex space E into a locally convex space F belongs to one of nine mutually exclusive and exhaustive classes or "states" according as the range of T is F, a proper dense subspace of F, or not dense in F, and according as T is injective and T^{-1} continuous, T is injective but T^{-1} is not continuous, or T is not injective. Let E, E^* and F, F^* be linear spaces in duality, and let T be a linear transformation from E into F which is continuous for the weak topologies $\sigma(E, E^*)$ and $\sigma(F, F^*)$. T then has an adjoint T^* which is continuous from F^* into E^* for the weak topologies $\sigma(F^*, F)$ and $\sigma(E^*, E)$. The author shows that of the 81 possible pairs of states the pair (T, T^*) might conceivably have, 72 are impossible, and 74 are impossible if E^* and F^* are the topological duals of E and F equipped with suitable locally convex topologies. The 9 remaining pairs of states do occur, as the author shows by examples of linear transformations on spaces of entire functions.

S. Warner (Durham, N.C.)

9833:

Ščel'nov, V. A. Many-valued linear operators in a locally convex space. Dokl. Akad. Nauk SSSR **132** (1960), 75–77 (Russian); translated as Soviet Math. Dokl. **1**, 510–512.

Let X and Y be separated locally-convex spaces with topological duals X^* and Y^*. A multi-valued linear operator from X to Y is essentially a linear subspace A of the vector space $X \times Y$, and its conjugate A^* is essentially the set of all $(y^*, x^*) \in Y^* \times X^*$ such that for all $(x, y) \in A$, $x^*(x) = y^*(y)$. {The author's definitions are quite obscure, but this is what he seems to have in mind.} Elementary topological relationships between A, A^*, their domains and ranges are given, and the sum and product of two multi-valued linear operators acting on a Fréchet space are briefly discussed; all results are straightforward extensions of well-known facts concerning single-valued linear operators defined on dense subspaces of a Hilbert space. *S. Warner* (Durham, N.C.)

9834:

Ėngel'son, Ya. L. [Engelsons, J.]. Some questions in the variational theory of non-linear equations in locally convex spaces. Latvijas Valsts Univ. Zinātn. Raksti **28** (1959), no. 4, 45–65. (Russian. Latvian summary)

First, some remarks on the background subject-matter of this article. A systematic account of the variational theory of non-linear equations is given in M. M. Vaĭnberg's textbook *Variacionnye metody issledovaniya nelineĭnyh operatov* [Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956; MR **19**, 567]. Let E and X be barreled (espaces tonnelés), locally convex, quasi-complete Hausdorff spaces; let E^* denote the strong dual of E. Suppose that f is a function on E to X whose Gâteaux-differential $df(x, h) = \lim_{t\to 0} t^{-1}[f(x+th) - f(x)]$ $(-1 < t < 1)$ exists for all x and h in E; if, moreover, the mapping $(\text{grd}\, f)(x) = \{h \to df(x, h)\}$ is a linear continuous mapping of E into X,

then f is called a "potential" on E, while grd f is called the gradient of f. A function F is called a "gradient" if F is a mapping of E into E^* that gives rise to a complex-valued potential f on E such that $F = \text{grd}\, f$; under these circumstances, f is known as the potential of F.

Henceforth, F is a gradient and A is a completely-continuous linear mapping of E^* into E. In its simplest form, the variational theory of non-linear operators of the form AF proceeds as follows: a study of the extremal points of the potential of F is transformed into conclusions concerning the existence and uniqueness of the solutions of the equation $u = AFu$.

Whereas Vaĭnberg's textbook deals only with the case where E is a Banach space, subsequent investigations of Vaĭnberg and Ėngel'son [Dokl. Akad. Nauk SSSR **122** (1958), 755–758; MR **21** #303] and Ėngel'son [Latvijas Valsts Univ. Zinātn. Raksti **20** (1958), no. 3, 27–45; MR **21** #826] have led to the setting of the present article, where E is not required to be a Banach space. However, the space E is subjected to another restriction, so as to appropriately generalize the case $E = L^p(\omega)$, where $1 < p < \infty$ and ω is a compact sub-interval of $(-\infty, \infty)$. Suppose for a moment that $E = L^p(\omega)$; investigations by V. V. Nemyckiĭ and M. Golomb (in the case $p = 2$) lead to the application of variational methods to the solution of the equation $u = AFu$. A particularly simple and relevant application is to the case where F is the Nemyckiĭ operator g_0^0 defined for all u in $L^p(\omega)$ by the equation $(g_0^0 u)(y) = g(u(y), y)$ (when $y \in \omega$); the prototypic equation is here the Hammerstein equation $u = AFu$, where $(AFu)(x) = \int_\omega (g_0^0 u)(y) \cdot K(x, y) \cdot dy$ (when $x \in \omega$).

The preceding paragraphs have sketched the setting of the article under review, which deals with non-linear equations of the type $u = AFu$. Now follows a sample result. By hypothesis, the barreled locally-convex vector space E gives rise to a Hilbert space H such that $E < H < E^*$ (here the notation $X < Y$ is used to indicate the existence of a continuous linear injection of X into a dense subset of Y); this hypothesis is satisfied by taking $H = L^2(-\infty, \infty)$ and $E = \{$all indefinitely-differentiable functions on $(-\infty, \infty)\}$. The completely-continuous linear mapping A (of E^* into E) is required to be such that its restriction A^H to H is both self-adjoint and positive-definite; let λ_1 be the largest eigenvector of A^H. As before, the gradient F is a mapping of E into E^*. Theorem: The equation $u = AFu$ has a unique solution $u \in E$, provided that F has a gradient $(\text{grd}\, F)(y) = \{s \to dF(y, s)\}$ such that $dF(y, s)(s) \geq 2 \cdot \lambda_1 \cdot \|s\|_H$ whenever $y, s \in E$. The author gives four theorems of the same nature as the preceding one; they differ chiefly in that the hypothesis on grd F is replaced by conditions on the potential of F.

The author concludes with the following applications, here presented in a simplified form. If $D(J)$ denotes the space of all indefinitely-differentiable complex-valued functions on J, let $K \in D(\omega \times \omega)$ be a symmetric kernel, and let $\{g_i : i = 1, \cdots, n\}$ be a family of continuous functions. Under a fairly simple additional assumption on the family $\{g_i : i = 1, \cdots, n\}$, there exists a function $u \in D(\omega)$ such that

$$u(x) = \int_\omega \sum_{i=0}^{n} [(g_i^n u)(y)] \cdot [(\partial/\partial y)^i K(x, y)] \cdot dy$$

for all x in ω, where ω is a compact sub-interval of $(-\infty, \infty)$, and $(g_i^n u)(y) = g_i(u(y), Du(y), \cdots, D^n u(y), y)$.

G. L. Krabbe (New Haven, Conn.)

9835:

Shibata, Toshio. Adjoint space and dual space. Proc. Japan Acad. **36** (1960), 261–266.

Suppose E is a separative locally convex linear space, E' is its dual space (all continuous linear functionals), and $\bar{E}$ is its adjoint space (all linear functionals bounded on bounded sets). The author discusses the relationship between E' and $\bar{E}$ and gives a number of interesting results; for example, E' and $\bar{E}$ coincide if and only if the Mackey topology on E is bornological.

D. C. Kleinecke (Berkeley, Calif.)

9836:

Nakano, Kazumi. On characterizations of projection operators. Proc. Japan Acad. **36** (1960), 196–197.

An elementary discussion about projection operators in vector lattices, the notion of which was introduced by H. Nakano [*Modulared semi-ordered linear spaces*, Maruzen, Tokyo, 1950; MR **12**, 420]. *I. G. Amemiya* (Tokyo)

9837:

Канторович, Л. В. [Kantorovič, L. V.]; Акилов, Г. П. [Akilov, G. P.]. ★Функциональный анализ в нормированных пространствах [Functional analysis in normed spaces]. Gosudarstv. Izdat. Fis.-Mat. Lit., Moscow, 1959. 684 pp. 23.05 r.

This treatise is designed for readers familiar with the theory of functions of a real variable and with linear algebra: there are no other formal prerequisites. The authors are particularly concerned with applications of functional analysis to the theory of approximation and the theory of existence and uniqueness of solutions of differential and integral equations (both linear and non-linear). This point of view has led them to include a great many specific examples of spaces and operators, some not readily accessible elsewhere, many involving intricate computations, and many of considerable interest. This emphasis on concrete applications is welcome and should make the book of wide usefulness.

Chapters I–V contain the elements of the theory of metric and topological spaces, normed spaces, linear operators and functionals, extension of linear functionals, and spaces of operators and functionals. The authors have done a sound and standard piece of work in this exposition. Chapter VI gives explicit forms of conjugate spaces of the usual elementary examples of Banach spaces, with a few novelties. Chapter VII begins with the Banach-Steinhaus theorem and draws a number of consequences from it, some of them quite concrete. Chapter VIII deals with weak convergence of functionals; it seems unfortunate that the authors limit themselves to weak sequential convergence instead of studying the weak topology in general. Chapter IX deals with completely continuous and adjoint operators, with particular attention to characterizing compact sets of various special spaces. Self-adjoint operators in Hilbert space are treated. The spectral theorem is proved in great detail. Chapter X deals with analytic representation of various classes of linear operators; for example, the bounded linear maps of l_p into l_r $(p > 1,\ r \geq 1)$ are identified. The analogous but more complicated results for L_p and L_r (over open sets of finite measure in Euclidean spaces) are also set forth in great detail. The chapter also contains the imbedding theorem of Sobolev [Mat. Sb. (N.S.) **2** (**44**) (1937), 467–500;

4 (46) (1938), 471–498]. Chapter XI deals with linear topological spaces in general; it is interesting to find barreled spaces making their appearance in a textbook for students.

Chapters XII through XVIII deal with functional equations, and comprise the applications of the preceding general theory, which are the authors' principal interest. Chapter XII deals with inversion of linear operators and is based on the well-known theorems of Banach. Chapter XIII treats special mappings of a Banach space into itself, these mappings being a generalization of the usual kernel operator. Chapter XIV describes the general theory of methods of approximation; chapter XV, the method of steepest descent; chapter XVI, the principle of invariant points; chapter XVII, differentiation of nonlinear operators; chapter XVIII, Newton's method.

Edwin Hewitt (Seattle, Wash.)

9838:

Slugin, S. N. Methods of successive approximations in almost ordered space. Bul. Inst. Politehn. Iaşi (N.S.) 4 (8) (1958), no. 3–4, 81–84. (Russian. English and Romanian summaries)

This note proves a theorem of the sort stated in the title and gives some examples of almost ordered spaces, that is, normed spaces in which the positive cone and norm are related by the condition that if a sequence is strictly decreasing and has limit zero then each term of the sequence is greater than zero. *M. M. Day* (Urbana, Ill.)

9839:

Gagliardo, Emilio. Interpolazione di spazi di Banach e applicazioni. Ricerche Mat. 9 (1960), 58–81.

Given two Banach spaces A and B, both contained (as vector spaces) in a vector space S, the author finds the following definition for "intermediate" Banach spaces. Given non-negative numbers $\alpha, \beta, \gamma, \delta$, with $\gamma+\delta=1$, an intermediate Banach space $(A, B)^{\alpha\beta\gamma\delta}$ is defined as the set of all $u \in S$ for which there exist a pair $\Gamma=(v, w)$ of functions $v(t)$ and $w(t)$ with the following properties: (a) $v(t)$ is defined for $t' \leq t < t''$, and takes values in A. It is continuous, $v(t')=0$ and its norm is continuous in every proper closed subinterval. (b) $w(t)$ is defined for $t' < t \leq t''$ and takes values in B. It is continuous, $w(t'')=0$ and its norm is continuous in every proper closed subinterval. (c) $v(t)+w(t)=u$ for $t' < t < t''$. (d) The following integral is finite:

$$\int_\Gamma \|v\|_A^\alpha \|w\|_B^\beta \|dv\|_A^\gamma \|dw\|_B^\delta =$$

$$\lim_{\substack{T' \to t' \\ T'' \to t''}} \lim_{\rho \to 0} \sum_{i}^{1,\cdots,n} \|v(\tau_i)\|_A^\alpha \|w(\tau_i)\|_B^\beta \times \|v(t_i)-v(t_{i-1})\|_A^\gamma \|w(t_i)-w(t_{i-1})\|_B^\delta$$

where $t' < T' = t_0 < \cdots < t_{i-1} \leq \tau_i \leq t_i < \cdots < t_n = T'' < t''$ and $\rho = \max |t_i - t_{i-1}|$.

In the space $(A, B)^{\alpha\beta\gamma\delta}$ one can introduce the norm

$$\phi_{\alpha\beta\gamma\delta}(u) = \inf_{\Gamma(u)} \left(\int_{\Gamma(u)} \|v\|_A^\alpha \|w\|_B^\beta \|dv\|_A^\gamma \|dw\|_B^\delta \right)^{1/(\alpha+\beta+1)}.$$

It is then fairly simple to prove that, given a linear operator T defined on a subspace of the vector space S and with range in a vector space S', and given Banach spaces A, $B \subset S$ and A', $B' \subset S'$, if $T: A \to A'$ is bounded with norm h and $T: B \to B'$ is bounded with norm k, then $T: (A, B)^{\alpha\beta\gamma\delta} \to (A', B')^{\alpha\beta\gamma\delta}$ is bounded with norm $h^{1-\theta}k^\theta$, where $\theta=(\beta+\delta)/(\alpha+\beta+1)$. Applications are made to obtain the classical interpolation theorems between L_p-spaces, to interpolation between L_p-spaces, with different measures (of E. Stein and G. Weiss), and to the W_p^r spaces of Sobolev. *G.-C. Rota* (Cambridge, Mass.)

9840:

Singer, Ivan. Les duals de certains espaces de Banach de champs de vecteurs. II. Bull. Sci. Math. (2) 83 (1959), 73–96.

Part I appeared in same Bull. (2) 82 (1958), 29–40 [MR 20 #3447]. Here, the author considers Banach spaces L^p_A $(1 \leq p < \infty)$ of (equivalence classes of) vector fields on a locally compact space Z whose pth power is integrable relative to a positive Radon measure on Z and to a fundamental family Λ of vector fields on Z. (Definitions are as in R. Godement, Ann. of Math. (2) 53 (1951), 68–124 [MR 12, 421].) He provides integral representations for the members of the dual spaces. The results subsume a variety of earlier generalizations of the classical representation theorems for such dual spaces.

M. Jerison (Lafayette, Ind.)

9841:

Singer, Ivan. Sur les espaces de Banach à base absolue, canoniquement équivalents à un dual d'espace de Banach. C. R. Acad. Sci. Paris 251 (1960), 2456–2458.

This note amends an earlier note with the same title [same C. R. 251 (1960), 620–621; MR 22 #5869]. In its amended form, the main result includes the equivalence of statements (d) and (f). If, in addition, a rearrangement of the basis is monotone, then these statements are equivalent to (c). *A. F. Ruston* (Sheffield)

9842:

Dinculeanu, Nicolae. Mesures vectorielles sur les espaces localement compacts. Bull. Math. Soc. Sci. Math. Phys. R. P. Roumaine (N.S.) 2 (50) (1958), 137–164.

This paper discusses a type of vector-valued measure from the point of view of the Bourbaki theory of integration. It is based in part on earlier work of the author, especially Compositio Math. 14 (1959), 1–22 and Proc. Amer. Math. Soc. 10 (1959), 59–68 [MR 21 #2908, 2909], but can be read without previous familiarity with this work. The outline of this paper is closely parallel to the scalar-valued theory in N. Bourbaki [*Éléments de mathématique*. XIII, Livre VI: Intégration, Chap. III, IV, Actualités Sci. Ind. No. 1175, Hermann, Paris, 1952; ibid. XXI, Chap. V, Actualités Sci. Ind. No. 1244, Hermann, Paris, 1956; MR 14, 960; 18, 881]. We now suggest in bare form the nature of the results. Let T be a locally compact space, let E, F be complex Banach spaces and let $\mathscr{K}_E(T)$ be the set of continuous maps of T into E which have compact supports. The author defines an (E, F) vector measure on T to be a linear mapping m of $\mathscr{K}_E(T)$ into F such that there is a positive Radon measure μ on T such that $|m(f)| \leq \mu(|f|)$ for all $f \in \mathscr{K}_E(T)$. (Here $|f|$ denotes the map $t \to |f(t)|$.) Multiplication of an (E, F) vector measure by a continuous or more general mapping of T into the space $\mathscr{L}(E)$ of bounded linear

operators on E is considered and corresponding "change of variables" theorems are found. If E, F are of denumerable type and F is the dual of Banach space, then forms of the Lebesgue-Nikodym theorem and the Lebesgue theorem are obtained. The vector measure induced by m on a subspace of T is considered. Finally, let A be a Banach algebra with unit element and consider those (A, A) vector measures m such that for all $a \in A$, $f \in \mathscr{X}_E(T)$, one has $m(af) = am(f)$ and $m(fa) = m(f)a$. For two such vector measures defined on two locally compact spaces T, T', the author defines a product measure on $T \times T'$ and obtains theorems of the Fubini type. *R. G. Bartle* (Urbana, Ill.)

9843:

Maslov, V. P. On some methods of functional analysis in the theory of operators and partial differential equations with parameters. Uspehi Mat. Nauk **14** (1959), no. 4 (88), 179–185. (Russian)

This paper proves several perturbation theorems for operators on a Hilbert space. For example, let $\{T_n\}$ be a sequence of pre-closed operators having a common domain, and let $T_n \to T$ in the strong operator sense. Suppose that x_0 satisfies $Tx_0 = 0$ and that there exist $S_n \subset T_n$ having uniformly bounded inverses. Then there exists a sequence $\{x_n\}$ such that $T_n x_n = 0$ and $x_n \to x_0$. Or again, let $T_n \to T$, and let $f_n \to f$ weakly. Suppose that there exists a bounded sequence $\{x_n\}$ with $T_n^* x_n = f_n$. Then there exists a sequence $\{y_n\}$ with $T^* y_n = f$ such that $x_n - y_n \to 0$ weakly.

For applications, the author treats the behavior as $\varepsilon \to 0$ of (1) the differential equation $\varepsilon^2 u_{tt} + a(x,t)u_t - \Delta u + u = f(x,t)$ with $x = (x_1, \cdots, x_n)$ and $u|_{t=0} = u_t|_{t=0} = u|_\Gamma = 0$, Γ being a sufficiently smooth contour in n-space; and (2) the ordinary differential equation $\varepsilon u^{(n)} + (a_k(x)u)^{(k)} + \cdots = f(x, \varepsilon)$. *G. Hufford* (Seattle, Wash.)

9844:

Putnam, C. R. A note on Toeplitz matrices and unitary equivalence. Boll. Un. Mat. Ital. (3) **15** (1960), 6–9.

Generalising previous results on unitary equivalence for the matrix $T = (c_{j-k})$ [Putnam, Pacific J. Math. **9** (1959), 837–846; MR **22** #183], it is shown that if $c_n = a_n e^{in\varphi}$ for all integral n, where a_n and φ are real, and $a_n = O(\alpha^n)$, $0 < \alpha < 1$, then there is a unitary matrix U such that $T = UGU^*$ where $G = (\int_0^\pi g(\theta) d\rho_{jk}(\theta))$, $g(\theta) \sim \sum_{-\infty}^{\infty} a_n e^{in\theta}$. *J. L. B. Cooper* (Cardiff)

9845:

Lyance, V. È. Conditions for closedness of the contraction of a self-adjoint operator. Dokl. Akad. Nauk SSSR **132** (1960), 1023–1026 (Russian); translated as Soviet Math. Dokl. **1**, 709–712.

Let Λ denote a self-adjoint operator with simple spectrum. Suppose Λ realized as multiplication by λ on a space $L_2(\sigma)$ where σ is a bounded monotone increasing function on the real axis, and denote by $\mathscr{D}_0$ the domain of Λ. The problem is to characterize the closed contractions of Λ. Let $M(\sigma)$ denote the linear space of all σ-measurable and σ-a.e. finite-valued functions on the real axis, and for a linear manifold $U \subset M(\sigma)$ denote by $U^\perp$ the collection of all functions $x \in L_2(\sigma)$ for which $\int_{-\infty}^{+\infty} u\bar{x}\, d\sigma = 0$ for all $u \in U$. Dually, for any linear manifold $V \subset L_2(\sigma)$ write $V^\perp$ for the collection of all functions $u \in M(\sigma)$ for which $\int_{-\infty}^{+\infty} v\bar{u}\, d\sigma = 0$ for all $v \in V$. (1) If A is a closed contraction of Λ with domain $\mathscr{D}$ then there is a linear manifold $U \subset M(\sigma)$ such that $\mathscr{D} = \mathscr{D}_0 \cap U^\perp$ (indeed, $U = (A^* - \tilde{\Lambda})\mathscr{D}_{A^*}$ will do, where $(\tilde{\Lambda}u)(\lambda) = \lambda u(\lambda)$ for all $u \in M(\sigma)$) and such that (α) $U \cap L_2(\sigma) = 0$ and (β) $\int_{-\infty}^{+\infty} |u(\lambda)|^2/(1+\lambda^2) d\sigma < \infty$ for all $u \in U$. Also, in the converse direction: (2) If U is a linear manifold in $M(\sigma)$ satisfying (α), (β) and if $(\mathscr{D}_0 \cap U^\perp)^\top = U$ then $\mathscr{D} = \mathscr{D}_0 \cap U^\perp$ is in fact the (dense by (α)) domain of a closed contraction of Λ.

The author points out that his arguments extend to direct sums of such multiplication operators. It seems difficult, however, to accede to his claim to an intrinsic characterization of the closed contractions of a self-adjoint operator, as the results obtained are all stated in terms of the canonical representation as multiplication on L_2. *A. Brown* (Houston, Tex.)

9846:

Markus, A. S. A basis of root vectors of a dissipative operator. Dokl. Akad. Nauk SSSR **132** (1960), 524–527 (Russian); translated as Soviet Math. Dokl. **1**, 599–602.

A sequence N_k of subspaces of a separable Hilbert space is a basis (for the closed linear hull of its union) if every x in the hull is uniquely expressed as $x = \sum_k x_k$ where x_k is in N_k. It is an orthogonal basis if all the N_k are orthogonal to one another; it is a Bari basis if there is an orthogonal basis such that $\sum_k \|P_k - Q_k\|^2 < \infty$ where P_k is the projection on N_k and Q_k is the projection on a member of the orthogonal basis. If λ is an eigenvalue of an operator A, then the corresponding root vectors are those such that $(\lambda I - A)^n x = 0$ for some n, and the root space is the set of all root vectors. The author announces conditions on the operator A and its eigenvalues for the root spaces to be a Bari basis, and for the existence of a Bari basis of root vectors. The conditions are especially useful if A is dissipative. *D. C. Kleinecke* (Berkeley, Calif.)

9847:

Brace, John W. Almost uniform convergence versus pointwise convergence. Proc. Amer. Math. Soc. **11** (1960), 986–987.

The author shows, in a slightly generalized form, the following fact. A uniform space E is compact if pointwise convergence and almost uniform convergence are equivalent in the space of all uniformly continuous real-valued functions on E [cf. Brace, Pacific J. Math. **9** (1959), 643–652; MR **22** #179]. *I. G. Amemiya* (Tokyo)

9848:

Ehrenpreis, L. Solution of some problems of division. IV. Invertible and elliptic operators. Amer. J. Math. **82** (1960), 522–588.

The usual notation of the theory of distributions is used. Let $S \in \mathscr{E}'$, and let J be the Fourier transform of S; the following conditions are equivalent: (1) S is invertible for $\mathscr{D}'$, i.e., $S * \mathscr{D}' = \mathscr{D}'$, (2) S is invertible for $\mathscr{E}$, i.e., $S * \mathscr{E} = \mathscr{E}$, (3) there exists an $a > 0$ such that for each x one can find a y such that $|x - y| \leq a \log(1 + |x|)$ and $J(y) \geq (a + |y|)^{-a}$. Several other equivalent conditions are given, e.g., $S * \mathscr{D}' \supset \mathscr{D}$. Some similar theorems hold for

$\mathscr{D}_F'$ (the space of distributions of finite order) instead of $\mathscr{D}'$ or $\mathscr{E}'$. Some partial results concerning the description of $S*\mathscr{D}'$ (S being not necessarily invertible) or the question of when $S*\mathscr{D}'\subset T*\mathscr{D}'$, are obtained.

Let $S\in\mathscr{E}'$ be invertible (for $\mathscr{D}'$) and consider the equation $S*f=0$, $f\in\mathscr{E}$; for any solution f there is an expansion in terms of exponential polynomial solutions.

Second main result: $\bigcap_{S\in\mathscr{E}'} S*\mathscr{D}'=\bigcap_{S\in\mathscr{E}'} S*\mathscr{E}=A$, where A denotes the space of all analytic functions on $\mathbf{R}^n$. The problem of whether $\bigcap_{S\in\mathscr{E}'} S*A=A$ is unsolved.

$T\in\mathscr{D}'$ is said to be C^∞ [resp. entire] in the variable x_1 (similar definition for a group of variables) if $\{a_j\partial^j/\partial x_1^j T\}$ is bounded in $\mathscr{D}'$ for a sequence of positive numbers a_j [resp. if $\{a^j/j!\,\partial^j/\partial x_1^j T\}$ is bounded in $\mathscr{D}'$ for any $a>0$]. Let $x=(x', x'', x''')$; $T\in\mathscr{D}'$ is said to be C^∞ [resp. entire] in x' relative to x'' if $T*(\delta(x',x'')h(x''))$ is C^∞ [resp. entire] in x' for any $h\in\mathscr{D}(x'')$. Now $S\in\mathscr{E}'$ is said to be C^∞ [resp. entire] elliptic in x' (or in x' relative to x'') if the following holds: if $S*U$ is C^∞ [resp. entire] in x' (or in x' relative to x'') then U has the same property. Taking $\mathscr{D}_F'$ instead of $\mathscr{D}'$ one gets corresponding concepts of weak ellipticities. The third main result of the paper is the characterization of elliptic S (for types of ellipticity described above) by means of some properties of zeros of the Fourier transform of S. If S is elliptic in all variables then S is invertible. Any S which is entire elliptic in x_1 has the form $S=\sum \partial^j\delta/\partial x_1^j(x_1-a)S_j(x_2,\cdots,x_n)$ for some number a and some finite sequence of distributions S_j.

An example of S is given such that the equation $S*T=\delta$ has no solution $T\in\mathscr{D}_M'$ for any $\mathscr{D}_M=\mathscr{D}\cap\mathscr{E}_M$ with a natural topology, $\mathscr{E}_M$ being Carlemann non-quasi-analytic. A list of unsolved problems is given.

S. Łojasiewicz (Kraków)

9849:

Vilenkin, N. Ya. On the theory of positive definite generalized kernels. Uspehi Mat. Nauk **15** (1960), no. 3 (93), 139–146. (Russian)

This article is concerned with the representations of positive definite generalized functions (distributions) defined on the spaces $S_{1/2}^{1/2}$ and $S_{1/2}$ of Gel'fand and Šilov. Suppose M is the space of points $z(z_1,\cdots,z_n)$, where the coordinates z_k are either real or purely imaginary. A representative result is the following: Suppose F is an even generalized function on $S_{1/2}^{1/2}$. In order that the inequality $(F,\phi*\phi^*)\geq 0$ hold for all even functions $\phi\in S_{1/2}^{1/2}$ it is necessary and sufficient that F be the Fourier transform of a uniquely determined even positive measure μ on M such that $\int_M e^{-c\omega^2}d\mu(z)<\infty$ for every $c>0$.

A. Devinatz (Princeton, N.J.)

9850:

Van Bladel, J. Distributions vectorielles. Ann. Télécommun. **15** (1960), 71–76.

Résumé de l'auteur: "La théorie des distributions de Schwartz a fourni une base théorique solide aux opérations que l'on effectue depuis de nombreuses années sur la fonction delta et ses dérivées. Le but de la présente note est d'étendre certains aspects de cette théorie aux opérateurs traditionnels du calcul vectoriel. Il s'agit ici d'un travail destiné aux ingénieurs, et dépourvu de la rigueur qu'un mathématicien pur serait en droit d'exiger (un exposé doué de toute la précision necessaire vient d être publié par L. Schwartz dans Ann. Inst. Fourier Grenoble **8** (1958), 1–209 [MR **22** #8322]). Son but est d'esquisser les idées principales de la théorie, tout en laissant dans l'ombre un énoncé précis des conditions (régularité des surfaces par exemple) dans lesquelles cette dernière est valable."

9851:

Duren, Peter L. Extension of a result of Beurling on invariant subspaces. Trans. Amer. Math. Soc. **99** (1961), 320–324.

In generalizing certain results of A. Beurling [Acta Math. **81** (1948); MR **10**, 381] the author considers the operator T: $y_n=\gamma x_{n-1}+\alpha x_n+\beta x_{n+1}$, $n=0,1,2,\cdots$; $x_{-1}=0$, where $\beta\neq 0$, $|\beta|\neq|\gamma|$. The eigenvalues of T are related to the Chebyshev polynomials of the second kind satisfying a certain orthogonality condition on the ellipse E: $\lambda=\beta e^{i\varphi}+\gamma e^{-i\varphi}$, $0\leq\varphi<2\pi$. Now let $f(\lambda)$ and $g(\lambda)$ be analytic inside of E such that $|f|^2$ and $|g|^2$ have a harmonic majorant there. We denote by F and G the analytic functions of z, $|z|<1$, arising from f and g by "transplantation" in the conformal mapping of the interior of E onto $|z|<1$. Then the following extension of one of the main results of Beurling is proved. The function g is in the space spanned by the functions $\lambda^n f(\lambda)$, $n=0,1,2,\cdots$, if and only if the "inner factor" of F divides the inner factor of G. (Concerning this concept see the paper of Beurling.)

G. Szegö (Stanford, Calif.)

9852:

Krasnosel'skiĭ, M. A. On a theorem of M. Riesz. Dokl. Akad. Nauk SSSR **131** (1960), 246–248 (Russian); translated as Soviet Math. Dokl. **1**, 229–231.

The author obtains the following extension of the convexity theorem of M. Riesz. Let $1\leq p_1, r_1, p_2, r_2\leq\infty$ be a set of indices and let A be a given linear operator which is continuous simultaneously as a mapping from L^{p_1} to L^{r_1} and L^{p_2} to L^{r_2}. Assume in addition that A is completely continuous as a mapping from L^{p_1} to L^{r_1}, and r_1 is finite. Then A is completely continuous as a mapping from L^p to L^r, where $1/p=t/p_1+(1-t)/p_2$, $1/r=t/r_1+(1-t)/r_2$, $0<t<1$. There is also a variant of the theorem for the case $r_1=\infty$.

E. M. Stein (Chicago, Ill.)

9853:

Talalyan, A. A. On series in bases of L_p space which are universal with respect to permutations. Akad. Nauk Armyan. SSR. Dokl. **28** (1959), 145–150. (Russian. Armenian summary)

Voir #9854, qui contient les mêmes résultats, avec démonstrations.

S. Mandelbrojt (Paris)

9854:

Talalyan, A. A. Series universal with respect to rearrangement. Izv. Akad. Nauk SSSR. Ser. Mat. **24** (1960), 567–604. (Russian)

(1) Soit $\{\varphi_n(x)\}$ un système de fonctions formant une base normée dans $L_p[0,1]$ $(p>1)$. Il existe une suite de nombres réels $\{a_n\}$, avec $\lim a_n=0$, tels que, quelle que soit la fonction mesurable $f(x)$ sur $[0,1]$, on peut réarranger les entiers $n\geq 1$, en une suite $\{\nu_k\}$ de telle sorte que $\sum a_{\nu_k}\varphi_{\nu_k}(x)$ converge en mesure vers $f(x)$ sur $[0,1]$. (2) Si $\{\varphi_n(x)\}$ est un système orthonormal de Haar, il existe une suite $\{a_n\}$

telle qu'à toute fonction mesurable, finie presque partout sur [0, 1], corresponde une suite d'entiers croissants $n_1<n_2<\cdots<n_k<\cdots$ telle que $\sum a_{n_k}\varphi_{n_k}(x)$ converge vers $f(x)$ presque partout sur [0, 1]. D'autres théorèmes, concernant, par exemple, les sommations par les moyennes de Cesàro, sont donnés. *S. Mandelbrojt* (Paris)

9855:

Rivkind, Ya. I. On a question in the theory of approximation of functions. Uspehi Mat. Nauk **14** (1959), no. 6 (90), 185–190. (Russian)

For a sequence of self-adjoint, completely continuous operators A_n which map $L^2(a, b)$ into itself, the inequality $\|A_n\varphi-\varphi\|\geqq K_\varphi\delta(n)$ is proved, where $K_\varphi>0$ for all $\varphi\in L^2$ except those approximable by the fixed points of the A_n and $\delta(n)=\min_k|1-\lambda_k^{(n)}|$, $\lambda_k^{(n)}$ running through all eigenvalues of A_n different from 1. This is used to obtain lower bounds for the degree of approximation by Fejér, de la Vallée-Poussin and other means, given recently by Butzer [Math. Z. **70** (1958), 93–112; Arch. Rational Mech. Anal. **5** (1960), 390–415; MR **20** #6039; **22** #8259], Sunouchi and Watari [Proc. Japan Acad. **34** (1958), 477–481; MR **20** #7183] and others. *G. G. Lorentz* (Syracuse, N.Y.)

9856:

Kreĭn, M. G. Criteria for completeness of a system of root vectors of a dissipative operator. Uspehi Mat. Nauk **14** (1959), no. 3 (87), 145–152. (Russian)

Let A be a bounded linear operator on a separable Hilbert space $\mathfrak{H}$; write $A=G+iH$, with G, H self-adjoint; and call A dissipative in case its imaginary part H is non-negative. A vector x in $\mathfrak{H}$ is a root vector of A if for some complex λ and some integer n one has $(A-\lambda I)^n x=0$. It is desired to obtain conditions assuring that the root vectors are complete in $\mathfrak{H}$. Henceforth suppose that A is completely continuous. Let $\mathrm{Sp}_+ G$ [resp. $\mathrm{Sp}_- G$] denote the sum of the positive [resp. negative] eigenvalues of G, counting multiplicities; if both are finite then G is said to have absolutely convergent trace. Extending some work of V. B. Lidskiĭ [Dokl. Akad. Nauk SSSR **115** (1957), 234–236; **119** (1958), 1088–1091; MR **20** #1221, 5430], the author gives a beautiful but non-elementary proof that the root vectors of $A=G+iH$ are complete in $\mathfrak{H}$ if $\mathrm{Sp}\,H$ is finite and if at least one of $\mathrm{Sp}_+ G$, $\mathrm{Sp}_- G$ is finite. The same conclusion is said to hold if G has absolutely convergent trace. *R. G. Bartle* (Urbana, Ill.)

9857:

Janoš, Ludvík. Homogeneous functionals on locally compact cones. Czechoslovak Math. J. **10** (**85**) (1960), 380–399. (Russian. English summary)

Soit L un espace vectoriel topologique et soit S un cône localement compact de L. Soit φ une fonctionnelle strictement positive, positivement homogène, sous-additive et continue définie sur S. Pour chaque nombre naturel n on pose $\varphi^n(x)=\sup\{\sum_1^n\varphi(y_i)|\ \sum_1^n y_i=x\}$. Alors $\psi(x)=\lim_n\varphi^n(x)$ est une fonctionnelle linéaire qui, sous certaines conditions, réalise la meilleure approximation de φ, au sens de la métrique

$$\rho(\varphi,\psi)=\log\{\max_{x\in S}[\varphi(x)/\psi(x)]\cdot\max_{x\in S}[\psi(x)/\varphi(x)]\}.$$

Comme application l'auteur considère l'équation intégrale $\int_0^1\Gamma(x,t)y(t)dM(t)=\lambda y(x)$, où le noyau $\Gamma(x,t)$ est continu, positif et positivement défini, et M est une mesure positive sur [0, 1]. La plus grande valeur propre $\lambda(M)$ de l'équation est une fonctionnelle positive, sous-additive et continue sur le cône S des mesures positives sur [0, 1], muni de la topologie faible. La meilleure approximation de $\lambda(M)$ est réalisé par la trace $\psi(x)=\int_0^1\Gamma(x,t)dM(t)$ de l'équation intégrale. *N. Dinculeanu* (Bucharest)

9858:

Lions, J. L. Théorèmes de trace et d'interpolation. I. Ann. Scuola Norm. Sup. Pisa (3) **13** (1959), 389–403.

An interpolation theorem is one which states that if a linear transformation is bounded between certain pairs of Banach spaces, then it is also bounded between certain other pairs. The first and most famous result of this kind is the theorem of M. Riesz (completed by Thorin) to the effect that if a linear transformation is bounded from L^{p_0} to L^{q_0} and from L^{p_1} to L^{q_1}, then it is also bounded from L^p to L^q whenever $(1/p, 1/q)$ is on the line segment joining $(1/p_0, 1/q_0)$ and $(1/p_1, 1/q_1)$. A trace theorem is one which characterizes the values at 0 of certain Banach space-valued functions on the half-line $t\geqq 0$. This paper gives a general construction for obtaining trace theorems and for proving interpolation theorems by means of trace theorems.

We begin with two of the interesting examples. (1) Every continuous linear transformation which takes C^0 into C^0 and C^1 into C^1 must also take C^α into C^α for $0<\alpha<1$. Here C^0 and C^1 are the spaces of continuous and continuously differentiable functions on the circle, and C^α is the space of Hölder-continuous functions with exponent α (Interpolation theorem). (2) Let $W^{1,p}(R^n)$ be the space of functions $u\in L^p$ with first derivatives also in L^p. The necessary and sufficient condition that a function f on a hyperplane R^{n-1} be the restriction of a function $u\in W^{1,p}(R^n)$ is that $f\in L^p(R^{n-1})$ and that

$$\int_{R^{n-1}}|y|^{-n-p+2}\int_{R^{n-1}}|f(x+y)-f(x)|^p\,dx\,dy<\infty$$

(Trace theorem). The paper actually contains more general results of this nature. As it is stated here, the result is due to Gagliardo.

The general constructions are made as follows. Let E be a complex Banach space and let Λ be a closed unbounded operator on E with domain D dense in E. Suppose that Λ is the infinitesimal generator of a bounded and strongly continuous semigroup $G(t)$, $t\geqq 0$. For $1<p\leqq\infty$ and $-1/p<\alpha<1-1/p$, let $E(p,\alpha,\Lambda)$ be the space of all $e\in E$ for which $\int_0^\infty\|t^{-1}\{G(t)e-e\}\|^p dt<\infty$, and let $W(p,\alpha,\Lambda)$ be the space of all functions $u(t)$ with values in D for which $\int_0^\infty t^{\alpha p}(\|u\|^p+\|\Lambda u\|^p+\|du/dt\|^p)dt<\infty$.

Trace theorem: The mapping $u\to u(0)$ maps $W(p,\alpha,\Lambda)$ continuously onto $E(p,\alpha,\Lambda)$.

Interpolation theorem: If π is a continuous linear transformation from E into E_1 and from D into D_1, then it is also a continuous linear transformation from $E(p,\alpha,\Lambda)$ into $E_1(p,\alpha,\Lambda_1)$.

The first example (which is due to the author but is not included in the paper) is obtained from the interpolation theorem by taking $E=C^0$, $G(t)$ to be the group of rotations, and $p=\infty$. The second example is obtained from the extension of the trace theorem to the case of several

commuting operators Λ_i by taking $E=L^p(R^{n-1})$, the Λ_i to be the $n-1$ first derivatives in R^{n-1}, and $\alpha=0$. Then $W(p, 0, \Lambda_1, \cdots, \Lambda_{n-1})$ can be interpreted as $W^{1,p}(R_+^n)$, R_+^n being one of the half-spaces determined by R^{n-1}.

K. T. Smith (Madison, Wis.)

9859:

Lions, Jacques-Louis. Une construction d'espaces d'interpolation. C. R. Acad. Sci. Paris **251** (1960), 1853–1855.

This paper gives a second general construction for obtaining interpolation theorems [see #9858].

Let A_0 and A_1 be Banach spaces contained in a topological vector space $\mathscr{S}$. Suppose that the identity mappings are continuous and that $A_0 \cap A_1$ is dense in A_0 and in A_1. Let $\mathscr{H}(A_0, A_1)$ be the space of functions $\zeta=\xi+i\eta \to f(\zeta) \in A_0+A_1$ which are holomorphic in $0<\xi<1$, are continuous and bounded in $0 \leq \xi \leq 1$, and satisfy $f(i\eta) \in A_0$ and $f(1+i\eta) \in A_1$ and $\|f\|_{\mathscr{H}}=\max[\sup_\eta \|f(i\eta)\|_{A_0}, \sup_\eta \|f(1+i\eta)\|_{A_1}]<\infty$.

Let ν be a measure on $[0, 1]$, and for $f \in \mathscr{H}$ put $Lf=\int_0^1 f(\xi)d\nu(\xi)$. Then L is a continuous linear transformation from $\mathscr{H}$ into A_0+A_1. Its range $[A_0, A_1; \nu]$ is a Banach space with the norm $\|a\|_{[A_0,A_1;\nu]}=\inf_{Lf=a}\|f\|_{\mathscr{H}}$.

Theorem: If π is a continuous linear transformation from A_0 into B_0 and from A_1 into B_1, then π is continuous from $[A_0, A_1; \nu]$ into $[B_0, B_1; \nu]$ of norm

$$\|\pi\|_\nu \leq \max(\|\pi\|_0, \|\pi\|_1).$$

If ν is the Dirac measure at a point $\theta \in]0, 1[$, then $\|\pi\|_\nu \leq \|\pi\|_0^{1-\theta}\|\pi\|_1^\theta$.

Some variations of this theorem are also given, and several interesting examples, concerning which we will mention only that if $A_0=L^{p_0}$ and $A_1=L^{p_1}$ and ν is the Dirac measure at θ, then $[A_0, A_1; \nu]=L^p$ with $1/p=(1-\theta)/p_0+\theta/p_1$.

K. T. Smith (Madison, Wis.)

9860:

Kreĭn, S. G. On an interpolation theorem in operator theory. Dokl. Akad. Nauk SSSR **130** (1960), 491–494 (Russian); translated as Soviet Math. Dokl. **1**, 61–64.

The author constructs an "analytical scale of spaces" in which norms are defined as the supremum on lines of complex-analytic vector-valued functions. He then goes on to prove abstract analogs and generalizations of the M. Riesz convexity theorem. Applications of these abstract theorems to various specific classes of spaces are given. The work parallels work of Calderón and of J.-L. Lions [#9859].

J. T. Schwartz (New York)

9861:

Gol'dengeršel', È. I. The spectrum of Volterra's operator on a half-axis and Tauber's theorems of the Paley-Wiener type. Dokl. Akad. Nauk SSSR **129** (1959), 971–974. (Russian)

The author continues his investigation of singular Volterra operators of the form $Vf=\int_0^x K(x, y)f(y)dy$ where K is a continuous (or, eventually, mildly discontinuous) $n \times n$ matrix-valued kernel on the octant $0 \leq y \leq x$ and f is a measurable and locally essentially bounded n-tuple on $[0, \infty)$. See same Dokl. **124** (1959), 1195–1198; **125** (1959), 19–22 [MR **21** #279; **22** #2866] for previous notations and results. Let α denote a continuous positive real-valued function on $[0, \infty)$ and let $Z(\alpha)$ denote the Banach space of functions f (as above) such that $f(x)\alpha(x) \to 0$ as $x \to \infty$, with norm $\|f\|_\alpha=\operatorname{ess\,sup}_{x \geq 0}\|f(x)\|\alpha(x)$. The problem posed here is to study the following Tauberian condition:

(T) If $(V-\lambda)f \in Z(\alpha)$ then $f \in Z(\alpha)$.

Proofs are not given, but the author indicates his method, which is to restrict V to be bounded on various spaces $Z(\alpha)$ and to employ his earlier spectral analyses of such operators. The results obtained are not readily summarized in a brief space but the following two main theorems are typical. Theorem 2: If V is bounded on $Z(\alpha)$ and if $\lim_{t\to\infty}\sup_{t \leq x}\int_t^x \|K(x, y)\|\alpha(x)/\alpha(y)\,dy<|\lambda|$ then (T) holds. Theorem 3: If $\tilde{K}$ is another kernel of the same type defining the operator $\tilde{V}$ and if V and $\tilde{V}$ are both bounded on $Z(\alpha)$ and if $\lim_{t\to\infty}\sup_{t \leq x}\int_t^x \|\tilde{K}(x, y)\|\alpha(x)/\alpha(y)\,dy=0$ then (T) holds for $V+\tilde{V}$ if and only if it holds for V.

A. Brown (Houston, Tex.)

9862:

Lidskiĭ, V. B. Summation of series over the principal vectors of non-selfadjoint operators. Dokl. Akad. Nauk SSSR **132** (1960), 275–278 (Russian); translated as Soviet Math. Dokl. **1**, 540–543.

If C is a completely continuous operator on a Hilbert space $\mathscr{H}$ with sufficiently many eigenvectors (e_s), every vector f in the space can be given a formal expansion $f \sim \sum c_s e_s$, where the coefficients (c_s) are determined by the eigenvectors of the adjoint operator C^*. Summability of the series expansion is obtained under a growth restriction which makes related entire functions be of finite order.

L. de Branges (Wayne, Pa.)

9863:

Lyubič, Yu. I. Almost periodic functions in the spectral analysis of operators. Dokl. Akad. Nauk SSSR **132** (1960), 518–520 (Russian); translated as Soviet Math. Dokl. **1**, 593–595.

Let A be a completely continuous (i.e., compact) operator in a weakly-complete normed linear space X; the operator A is subjected to the following condition:

$$\sup_{-\infty<\lambda<\infty}\|e^{tA\lambda}\| < \infty.$$

Under these circumstances, $p(A) \subset (-\infty, \infty)$, where $p(A)$ is the set of all eigenvalues of A. It is possible to define an orthogonal family $\{P_\lambda: -\infty<\lambda<\infty\}$ of uniformly bounded linear projectors such that $AP_\lambda=P_\lambda A=\lambda \cdot P_\lambda$ and

$$P_\lambda = \operatorname*{weak\,limit}_{n\to\infty}(2n)^{-1}\int_{-n}^{n} e^{tAt}\cdot e^{-\lambda t}\,dt;$$

furthermore, P_λ is a projection of X onto the eigenmanifold $\{x \in X: Ax=\lambda \cdot x\}$. The set of all eigenvectors forms a complete (i.e., fundamental) subset of X. Suppose that $x \in X$. Various meanings are given to the following formal relations:

$$x \sim \sum_{\lambda=-\infty}^{\infty} P_\lambda x, \quad \text{and} \quad Ax \sim \sum_{\lambda=-\infty}^{\infty} \lambda \cdot P_\lambda x$$

(of course, $P_\lambda x=0$ when λ is outside the denumerable set $p(A)$). For example, the series on the left-hand side is strongly summable to x by means of a certain (Fejér-Bochner type) summability method.

The proofs are essentially dependent on the theory of almost-periodic functions and its attendant theorems on Fourier series. *G. L. Krabbe* (New Haven, Conn.)

9864:

Heuser, Harro. Zur Eigenwerttheorie einer Klasse symmetrischer Operatoren. Math. Z. **74** (1960), 167–185.

The author extends numerous results about the spectral properties of symmetric finite operators, which he has dealt with in a recent paper [Trans. Amer. Math. Soc. **94** (1960), 327–336; MR **22** #2903] to a more general situation. Let $\mathfrak{S}$ be a pre-Hilbert space (not necessarily either separable or complete). Interest is focussed on those linear transformations A of $\mathfrak{S}$ into itself possessing the property (E): If the complex number $\lambda \neq 0$ is not an eigenvalue of A, then $A-\lambda I$ maps $\mathfrak{S}$ onto itself. (E) is fulfilled whenever A is a symmetric finite operator. The basic technique employed makes use of the imbedding of $\mathfrak{S}$ in its completion $\overline{\mathfrak{S}}$ and using the unique symmetric closed extension A^* of A; this A^* is self-adjoint and bounded. Known spectral properties of A^* are shown to imply similar results about A. In particular, information is obtained concerning the possibility of expanding each element of the range of A in a series of orthonormal ordinary or "generalised" eigenvectors of A belonging to non-zero eigenvalues. These generalised eigenvectors of A belonging to λ are the elements e of $\overline{\mathfrak{S}}$ which are limits in $\overline{\mathfrak{S}}$ of sequences (s_n) extracted from $\mathfrak{S}$ such that $(A-\lambda I)s_n \to 0$. The detailed results are too numerous to record here. *R. E. Edwards* (Reading)

9865:

Koppelman, Walter. On the spectral theory of singular integral operators. Trans. Amer. Math. Soc. **97** (1960), 35–63.

This paper concerns special singular integral operators of the form

$$Lx(\lambda) = f(\lambda)x(\lambda)+\frac{1}{\pi i}\int_a^b \frac{(k(\lambda)k(\mu))^{1/2}}{\mu-\lambda}\,x(\mu)\,d\mu,$$

with certain technical restrictions on f and k. The domain of L is to be $L^2(a, b)$. The author obtains first some information on the spectrum of the singular part of L, that is,

$$\hat{L}x(\lambda) = f(\lambda)x(\lambda)+\frac{k(\lambda)}{\pi i}\int_a^b \frac{x(\mu)}{\mu-\lambda}\,d\mu,$$

which differs from L by a completely continuous operator. The methods are function-theoretic and follow those of Carleman. One result is the following. Theorem: If ξ is not on the closed interval

$$[\min_{a\le\lambda\le b}\{f(\lambda)-k(\lambda)\},\ \max_{a\le\lambda\le b}\{f(\lambda)+k(\lambda)\}]$$

then ξ is in the resolvent set of L.

A detailed study of the singular operator $\hat{L}$ enables the author to establish the existence of isometries R and S, such that $RS=SR=I$, of a subspace L_Δ of $L^2(a, b)$ onto $L^2(\min\{f(\lambda)-k(\lambda)\},\ \max\{f(\lambda)+k(\lambda)\})$. Furthermore, L_Δ is invariant under $\hat{L}$ and if $h(\lambda)\in L_\Delta$ and $g(\xi)=Rh(\lambda)$ then $\xi g(\xi)=RLh(\lambda)$.

A further restriction on the coefficients of the equation, expressed in a rather complicated way in terms of the inverse of the operator L, leads to some sharper results. For example, it is shown that there exists an isometry $\mathfrak{R}$ of $L^2(a, b)$ itself into the direct sum of a finite number of L^2 spaces with $\mathfrak{R}$ coinciding with the previous transformation R on L.

Simple criteria are given in terms of f and k for the operator L to have a simple continuous spectrum from $\min\{f(\lambda)-k(\lambda)\}$ to $\max\{f(\lambda)+k(\lambda)\}$ or for a continuous spectrum of multiplicity less than or equal to 2.

A few remarks are included on the more general singular operator,

$$Mx(\lambda) = f(\lambda)x(\lambda)+\frac{1}{\pi i}\int_a^b \frac{g(\lambda,\mu)}{\mu-\lambda}\,x(\mu)\,d\mu.$$

R. C. MacCamy (Pittsburgh, Pa.)

9866:

Krasnosel'skiĭ, M. A. Solution of equations involving adjoint operators by successive approximations. Uspehi Mat. Nauk **15** (1960), no. 3 (93), 161–165. (Russian)

The main theorem is the following: If A is a self-adjoint (but not necessarily completely continuous) operator in Hilbert space for which $\|A\|\le 1$, but -1 is not an eigenvalue, and if $\varphi = A\varphi+f$ has a solution, then the sequence of iterates $\varphi_n = A\varphi_{n-1}+f$ has a limit which is a solution. Application is made to some special cases.

A. S. Householder (Oak Ridge, Tenn.)

9867:

Putnam, C. R. Group commutators of bounded operators in Hilbert space. Michigan Math. J. **7** (1960), 229–232.

Let A be a unitary, B a nonsingular (B is bounded and has a bounded two-sided inverse B^{-1}) operator on Hilbert space. For the commutator $D=ABA^{-1}B^{-1}$ it is assumed that (*) $AD=DA$. With these hypotheses, the following conclusion is reached: Either (i) $\mathrm{sp}(D)\equiv$ spectrum of $D=1$, or (ii) $0\in W(B)$ or (iii) $0\in W(B^{-1})$ (where $W(H)$ is the closure of the set of numbers $\{(Hx, x)|\ \|x\|=1\}$ for any (bounded) operator H). Corollaries: (1) Either $AB=BA$ or $0\in W(B)$. (2) If, in addition to (*), D is assumed normal, then $AB=BA$ or $0\in W(B)\cap W(B^{-1})$.

Among alternative approaches in this area, the following is offered: Assume (*) and that either D^nB is normal for $n=0, 1, 2, \cdots$, or for $n=0, -1, -2, \cdots$. Then either $\mathrm{sp}(D)=1$ or $0\in W(B)\cap W(B^{-1})$.

The basic tool is the relation $D^nB=A^nBA^{-n}$, $n=0, 1, 2, \cdots$, derived from (*).

B. R. Gelbaum (Princeton, N.J.)

9868:

Hsu, Leetsch C. An estimation for the first exponential formula in the theory of semi-groups of linear operations. Czechoslovak Math. J. **10** (85) (1960), 323–328. (Russian summary)

Let $\{T(\xi), \xi>0\}$ be a strongly continuous semi-group of bounded operators on a complex Banach space X. Let $A_\eta=\eta^{-1}[T(\eta)-I]$. The "first exponential formula" [see E. Hille, *Functional analysis and semi-groups*, Amer. Math. Soc. Colloq. Publ. Vol. 31, New York, 1948; MR **9**, 594] then states that for each $x\in X$ the quantity

$$\|\exp[(\xi-\alpha)A_\eta]T(\alpha)x-T(\xi)x\|$$

approaches 0 with η. The present paper sharpens this result by showing that for small enough η the above

expression is bounded by $\mu(\eta^{1/3}, x)+K\eta^{1/3}\|x\|$, where $\mu(\delta, x)$ is the rectified modulus of continuity of $T(\xi)x$ and K is independent of both η and x. This estimate is obtained as a corollary to a similar estimate for the singular series $e^{-st}\sum_k (st)^k f(k/t)/k!$, where f is a continuous function satisfying $|f(s)| \leq M^{1+s}$ for all $s \geq 0$. Also established are several interesting polynomial approximations.

G. Hufford (Seattle, Wash.)

9869:

Bonsall, F. F. Positive operators compact in an auxiliary topology. Pacific J. Math. **10** (1960), 1131–1138.

The author develops some ideas previously used by himself [Proc. London Math. Soc. (3) **8** (1958), 53–75; MR **19**, 1183] and by Schaefer [Pacific J. Math. **10** (1960), 1009–1019; MR **22** #5893] and obtains a further extension of the Krein-Rutman theorem and related theorems. The basic idea is to introduce, in a partially ordered normed linear space, an auxiliary topology suitably related to the norm topology, and to exploit compactness relative to this auxiliary topology. When the auxiliary topology coincides with the norm topology, the main theorem reduces to Theorem 1 of the author's paper cited above. Proofs are technically elementary.

J. H. Williamson (Cambridge, England)

9870:

Gårding, Lars. Vecteurs analytiques dans les représentations des groups de Lie. Bull. Soc. Math. France **88** (1960), 73–93.

In 1947 the author showed [Proc. Nat. Acad. Sci. U.S.A. **33** (1947), 331–332; MR **9**, 133] that given a strongly continuous representation $x \to R(x)$ of a Lie group G on a Banach space B the set of C^∞ vectors (elements a of B such that the mapping $x \to R(x)a$ of G into B is C^∞) is dense in B, because if f_n is a sequence of C^∞ functions with compact support in G approaching the Dirac measure at the identity, then $\int f_n(y^{-1})R(y)a\omega(y)$ (where ω is left-invariant Haar measure) is a C^∞ vector tending to a as $n \to \infty$. The vector a is called an analytic vector in case the mapping $x \to R(x)a$ of G into B is (real) analytic. If analytic functions f_n can be found which tend to the Dirac measure at the identity and which decrease sufficiently rapidly at infinity for the integral to be defined, the same argument shows that the analytic vectors are dense. In § 8 of the reviewer's paper Ann. of Math. (2) **70** (1959), 572–615 [MR **21** #5901] such functions were constructed using the heat equation, some probability theory, and some helpful advice from the present author. The result is that the analytic vectors are dense for any strongly continuous representation of a Lie group on a Banach space. Previous results were obtained by different methods by Harish-Chandra and by Cartier and Dixmier. In this paper the author gives a complete treatment of the question in which the rapid decrease of the fundamental solution of the heat equation is shown by a direct analytical argument used by Gaffney (who also expresses his indebtedness to the author) in Comm. Pure Appl. Math. **12** (1959), 1–11 [MR **21** #892].

E. Nelson (Princeton, N.J.)

9871:

Maurin, K. Bemerkungen über das Spektrum und Eigenfunktionen (links) invarianter Differentialoperatoren auf Lieschen Gruppen. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astr. Phys. **8** (1960), 3–9. (Russian summary, unbound summary)

The author derives spectral properties of operators in the left-invariant enveloping algebra of a Lie algebra of a Lie group by observing that they commute with an elliptic right-invariant operator. He considers both the regular representation and operators on open subsets of the group. His proofs are incomplete due to the assumption that commutativity of two unbounded operators on their domain implies commutativity in the sense of spectral theory. In the case of the regular representation (Theorem 2) this difficulty may be avoided by using the results of Stinespring and the reviewer [Amer. J. Math. **81** (1959), 547–560; MR **22** #907]. The author's Theorem 4 is incorrect, as is shown by the example in which $G = R^2$, Ω is the open unit disc, B_Ω is the Green's operator corresponding to the Dirichlet problem and P_V is the Laplace operator corresponding to the Neumann problem.

E. Nelson (Princeton, N.J.)

9872:

Ghelfand, I. M.; Peateĭki-Şapiro, I. I. = Pyateckiĭ-Sapiro, I. I. Theory of representations of groups and theory of automorphic functions. Acad. R. P. Romîne An. Romîno-Soviet. Ser. Mat.-Fiz. (3) **14** (1960), no. 2 (33), 60–84. (Romanian)

Translation of a Russian original [Uspehi Mat. Nauk **14** (1959), no. 2 (86), 171–194; MR **22** #2912].

9873:

Levitan, B. M. Lie theorems for generalized operators of translation. Dokl. Akad. Nauk SSSR **123** (1958), 32–35. (Russian)

Let V_n be a differentiable manifold and L a linear space of functions on V_n. To each s in V_n let there correspond a linear operator T^s on L. The author uses the notation $T_t^s f(r, t)$ to denote the action of T^s on f as a function of t keeping r constant. The collection of such T^s is a "generalized translation operator" [compare the author's previous paper, Uspehi Mat. Nauk **4** (1949), no. 1 (29), 3–112; MR **11**, 116] if (1) there exists a neutral element s_0 such that T^{s_0} is the identity; (2) there exists a subspace $M \subset L$ such that $T_t^s f(t)|_{t=s_0} = f(s)$ for all s in V_n, f in M; and (3) the law of associativity $T_{t'}^{s'} T_t^s f(t) = T_{t'}^{s} T_t^{s'} f(t)$ holds. For example, if V_n is a Lie group we may set $T_t^s f(t) = f(ts)$. The present paper announces several results concerning the infinitesimal behavior at s_0 of such a generalized translation operator. Particular attention is paid to the case where $T_t^s f(t) = \int_V f(u) d_u \sigma(s, t, u)$, and the announced results seem quite analogous to the elementary statements concerning the structure constants of a Lie group.

G. Hufford (Seattle, Wash.)

9874:

Levitan, B. M. Converse Lie theorems for general operators of generalized translation. Dokl. Akad. Nauk SSSR **123** (1958), 243–245. (Russian)

This paper is a continuation of the preceding. Here it is assumed that V_n and all functions involved are analytic. The theorems announced are to the effect that under certain conditions the structure constants uniquely determine, at least locally, a generalized translation operator.

G. Hufford (Seattle, Wash.)

9875:

Levitan, B. M. Adjoint operators of generalized translation. Dokl. Akad. Nauk SSSR **123** (1958), 401–404. (Russian)

Using the terminology and notation of the preceding two papers, the author now introduces an L^2 space of functions on V_n and considers the operators $\hat{T}^t$ which are adjoint to the generalized translation operator T^t. Two cases seem of interest to the author: the "noncommutative case" in which $\hat{T}^\tau T^t f(t) = T^t \hat{T}^\tau f(t)$, and the "commutative case" in which $\hat{T}^\tau T^t f(t) = T^\tau \hat{T}^t f(t)$. Conditions on the related infinitesimal operators are derived, at least formally, which insure the noncommutative case.

G. Hufford (Seattle, Wash.)

9876:

Gel'fand, I. M.; Graev, M. I. Fourier transforms of rapidly decreasing functions on complex semisimple groups. Dokl. Akad. Nauk SSSR **131** (1960), 496–499 (Russian); translated as Soviet Math. Dokl. **1**, 276–279.

Let G be a complex semi-simple Lie group and Γ the ring of all rapidly decreasing (i.e., $o(\|l(g)\|)$ for all finite-dimensional representations l of G) indefinitely differentiable functions on G. The authors describe the image of Γ under the Fourier transform on G. This generalizes a previous result [same Dokl. **124** (1959), 19–21; MR **22** #3987] of the first-named author from $SL(2, C)$ to arbitrary complex semi-simple Lie groups. The case of $SL(2, R)$ was treated by L. Ehrenpreis and the reviewer [Ann. of Math. (2) **61** (1955), 406–439; Trans. Amer. Math. Soc. **84** (1957), 1–55; **90** (1959), 431–484; MR **16**, 1017; **18**, 745; **21** #1541].

For each character χ of a fixed Cartan subgroup there exists a unitary representation U_χ of G which can be described in a Hilbert space of functions of z, where z varies over the subgroup generated by the positive roots. This is the well-known principal series, which is irreducible for almost all χ. For $f \in \Gamma$, the operator $\int f(g) U_\chi(g) dg$ has an integral kernel $K(z_1, z_2, \chi)$. The image of Γ is described by means of conditions on these functions $K(z_1, z_2, \chi)$. Some of these conditions are growth and regularity conditions on K; others come from the fact that if χ_1 and χ_2 go over into each other under the Weyl group their U_{χ_1} and U_{χ_2} are equivalent, and for such χ_1, χ_2 one has a condition of the form $BK(z_1, z_2, \chi_1) = K(z_1, z_2, \chi_2)B$. There are, however, further conditions on the K's, called singular relations by the authors. These singular relations correspond to singular points in the space of characters χ and seem closely connected to characters which are fixed points under elements of the Weyl group and to the singular relations encountered by F. Bruhat [Bull. Soc. Math. France **84** (1956), 97–205; MR **18**, 907].

F. I. Mautner (Baltimore, Md.)

9877:

Tsuji, Kazô. General crossnorms. Proc. Japan Acad. **36** (1960), 415–417.

In his treatise on tensor products of Banach spaces, Schatten constructs a "self-associate crossnorm" [*A theory of cross-spaces*, Princeton Univ. Press, Princeton, N.J., 1950; MR **12**, 186; Appendix II; see also Trans. Amer. Math. Soc. **53** (1943), 195–217; MR **4**, 161]. The author of the note under review generalizes this construction by replacing the arithmetic mean by more general weighted means, and so constructs two "general" crossnorms which are associates of each other.

{Reviewer's comment: As in the parent work, the same notation α' is used for the crossnorm on $\mathbf{B}_1^* \odot \mathbf{B}_2^*$ associated with a given crossnorm α on $\mathbf{B}_1 \odot \mathbf{B}_2$ and for the crossnorm on $\mathbf{B}_1 \odot \mathbf{B}_2$ associated with a given crossnorm α on $\mathbf{B}_1^* \odot \mathbf{B}_2^*$. That this confusion does not invalidate the construction follows from a paper by the reviewer [J. London Math. Soc. **32** (1957), 110–112; MR **19**, 565].}

A. F. Ruston (Sheffield)

9878:

Schatten, Robert. ★Norm ideals of completely continuous operators. Ergebnisse der Mathematik und ihrer Grenzgebiete. N. F., Heft 27. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1960. vii+81 pp. DM 23.60.

This volume contains the greater part of the theory of completely continuous operators and of crossnorms of operators on a unitary space in the author's previous book, *A theory of cross-spaces* [Princeton Univ. Press, Princeton, N.J., 1950; MR **12**, 186]. But certain extensions are clarified, and also some closely related theorems due to other authors are stated, which were not included in the above book. The theory of crossnorms on ideals of completely continuous operators on a Hilbert space H is the main development. The proofs in this are concise, but very clearly written.

Let $\mathscr{A}$ be the full algebra of bounded operators on H, and let C and R be the subalgebras of $\mathscr{A}$ consisting of all completely continuous or finite-rank operators, respectively, on H. Denote the bound of an operator A ($\in \mathscr{A}$) by $\|A\|$.

In chapter I, many fundamental known theorems are given, among these being theorems relative to the spectrums of completely continuous operators and theorems on the ideals of such operators. In particular, a theorem on an absolutely maximal non-trivial ideal in $\mathscr{A}$ is given, together with a theorem of Calkin relative to characteristic set of ideals ($\subset C$) and also a theorem of Kaplansky relative to uniformly closed left ideals ($\subset C$).

In chapter II, the author discusses the operators of the Schmidt-class (σc) and the Schmidt-norm σ and, in chapter III, operators of the trace-class (τc) and trace-norm τ (τ was denoted by m in the previous book). In particular, the author states a characterization of an integral-type operator over L^2-space by an operator in (σc), and a maximum problem in (τc) is proved: Let $t(A)$ be the trace of $A \in (\tau c)$ and $\mathscr{R}(c)$ the real part of the complex number c; then $\mathscr{R}t(A) \geq \mathscr{R}t(UA)$ for all unitary U if and only if $A \geq 0$, and the following three conditions are equivalent: U_0 is maximal for $\mathscr{R}t(UA)$, $U_0 A = [A]$ $(= (A^*A)^{1/2})$, and $\mathscr{R}t(U_0 A) = \tau(A)$. This is applied to the maximum problem for n-dimensional unitary space H_n in the last chapter.

In chapter IV, consider $\mathscr{A}$ and C as Banach spaces with $\|A\|$ as their norms; then the following theorems are described: $C^* = (\tau c)$ and $C^{**} = \mathscr{A}$, i.e., the first relation indicating that there is a one-one correspondence $f \in C^* \leftrightarrow T_f \in (\tau c)$ such that $f(A) = t(T_f A)$ for all $A \in C$ and $\|f\| = \tau(T_f)$ and the second indicating a similar correspondence. According to a result of Dixmier, every $f \in \mathscr{A}^*$ is uniquely represented in the form $f = f_1 + f_2$ by $f_1 \in (\tau c)$ and $f_2 \in C^\perp$, and $\|f\| = \|f_1\| + \|f_2\|$. Using this, the author proves that if $f \in \mathscr{A}^*$ is strongest continuous then f is precisely in (τc) and conversely. As the author states, general modifications to the arguments of the above theorems have

been already proved by several authors. (A related book of Dixmier should be mentioned here: *Les algèbres d'opérateurs dans l'espace hilbertien* [Gauthier-Villars, Paris, 1957; MR **20** #1234].)

Chapter V is concerned with the norm ideal, and this is a main part of this volume. Let I be an ideal in $\mathscr{A}$. A norm $\alpha=\alpha(A)$ on I is termed a crossnorm if it possesses the cross-property "$\alpha(A)=\|A\|$ for $A\in I$ of rank 1". α is termed unitarily invariant if $\alpha(UAV^*)=\alpha(A)$ for $A\in I$ and for any pair of unitary U, V, and α is termed uniform, if $\alpha(XAY)\leq\|X\|\,\|Y\|\alpha(A)$ for any $A\in I$ and for $X, Y\in\mathscr{A}$. For a crossnorm α on R, a non-negative function α' on R is defined such that $\alpha'(A)=\sup\{|t(XA)|/\alpha(X);\ 0\neq X\in R\}$ for $A\in R$. Then the function α' has the following properties: $\alpha'(X)\geq\|X\|$ for all $X\in R$ if and only if α' is a crossnorm on R, and in this case α'', α''', $\cdots$ are also crossnorms on R. Denote by $R(\alpha)$ the linear space R normed by a crossnorm α with $\alpha(X)\geq\|X\|$ for all $X\in R$, and by R_α the metric completion of $R(\alpha)$ which is identified with a Banach space of operators in C. After developing the various calculi of the unitarily invariant crossnorms of the linear space of all operators on H_n and the symmetric gauge functions on n-dimensional space of n-tuples of real numbers, the author shows that such crossnorms on R and such a gauge function on L generate each other, where L denotes the set of all infinite sequences of real numbers having only a finite number of non-zero terms, and also shows that unitary invariance and uniformity of crossnorm α on R are equivalent. Moreover, he shows that for such α, $\alpha''=\alpha$ holds, and putting $\alpha_1=(\alpha+\alpha')/2$, $\alpha_n=(\alpha_{n-1}+\alpha'_{n-1})/2$ $(n>1)$ that $\alpha_n\downarrow\alpha$ holds. (These results were already given in the author's previous book, where R, $R(\alpha)$ and R_α were denoted by $H\odot\bar{H}$, $H\odot_\alpha\bar{H}$ and $H\otimes_\alpha\bar{H}$, respectively, but where the crossnorm α was only defined on $H\odot\bar{H}$.)

An ideal $I\subset\mathscr{A}$ is termed a norm ideal if on it there is defined a uniform crossnorm with respect to which I is also complete. A norm ideal I is minimal if none of its proper subspaces is also a norm ideal (with the same crossnorm). The author defines, similarly to α', $\|A\|_s=\sup\{|t(XA)|/\alpha(X);\ 0\neq X\in R\}$ for $A\in\mathscr{A}$. If $\|A\|_\alpha<+\infty$, A is termed of finite α-norm; then $\alpha'(A)=\|A\|_s$ for $A\in R$, and $\|A\|\leq\|A\|_s\leq\tau(A)$ for any A. In particular, the Schmidt norm σ and the trace norm τ are unitarily invariant, and $(\sigma c)=R_\sigma=R_{\sigma'}=R_\sigma{}^*$, $(\tau c)=R_\tau$, and $\|A\|_\tau=\|A\|$ for all $A\in\mathscr{A}$. The author also states, relative to $\|\cdot\|_\alpha$, that the set of all operators A with $\|A\|_\alpha<+\infty$ is a Banach space in which the norm is defined by $\|A\|_\alpha$, and that it is equivalent to $R_\alpha{}^*$, in the sense of the theorem of $C^*=R_\tau=(\tau c)$ as described in the previous part.

In the final two sections, the following are proved. The Banach space $R_\alpha{}^*$ is a norm ideal if and only if R_α is a minimal norm ideal and this is the case if and only if α is unitarily invariant. Every minimal norm ideal I is characterized by an R_α for some unitarily invariant α. For a unitarily invariant α, the following are equivalent: R_α is reflexive, $R_{\alpha'}$ is reflexive, both $R_\alpha{}^*$ and $R_{\alpha'}{}^*$ are minimal. The author also states, using theorems due to Rickart, that for a unitarily invariant α the norm ideals R_α and $R_\alpha{}^*$ admit essentially unique norms $\alpha(A)$ and $\|A\|_s$ respectively. In general, $R_\alpha{}^*\supset R_{\alpha'}$ holds for any unitarily invariant α. Relative to this inclusion, he sets up an open problem: Is there a direct characterization of unitarily invariant crossnorms α for which $R_\alpha{}^*=R_{\alpha'}$ holds?

H. Umegaki (Tokyo)

9879:

Takesaki, Masamichi. On the non-separability of singular representation of operator algebra. Kōdai Math. Sem. Rep. **12** (1960), 102–108.

Let M be a von Neumann algebra whose type I finite part is a finite sum of factors. Let π be a $*$ homomorphism of M into bounded operators on the separable Hilbert space H. It had been conjectured by J. Feldman and J. M. G. Fell that π is automatically σ-weakly continuous, and this was in fact shown provided also the type II finite part was a finite sum of factors. This paper now handles the general type II finite case, proving the conjecture completely. The author works with states rather than homomorphisms, making use of theorems about states which he has developed in previous papers [Proc. Japan Acad. **35** (1959), 365–366; MR **22** #3991].

J. Feldman (Berkeley, Calif.)

9880:

Mibu, Yoshimichi. On the theory of regular functions in Banach algebras. Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math. **33** (1960/61), 323–340.

The author brings significant advances to the theory initiated by E. R. Lorch [Trans. Amer. Math. Soc. **54** (1943), 414–425; MR **5**, 100] of analytic functions in a Banach algebra B. His principal results concern an open-mapping theorem and the analyticity of the inverse function of $w=f(z)$ where f is analytic over the entire algebra B. In order to obtain these theorems, the author sharpens some of the early results due to the reviewer. He begins by a consideration of the radius of convergence of power series. Let $f(z)$ be analytic in an open set $D\subset B$ and let $a\in D$. Let $\rho(a, D)$ denote the radius of the maximal open sphere in D with center a. After some six preliminary results the author proves that the known expansion formula: $f(z)=\sum_{n=0}^{\infty}f^{(n)}(a)(z-a)^n/n!$ is valid for the entire sphere $\|z-a\|<\rho(a, D)$. Next he considers the open mapping phenomenon. He is interested in a point a lying in the domain of analyticity of $f(z)$ and such that $[f'(a)]^{-1}$ exists. He proves then that $f(D)$ contains a neighborhood of $f(a)$. The argument is straightforward but the inequalities are delicate. Next, he shows that there exists a neighborhood U of a such that f is one-to-one on U. Let $z=\varphi(w)$ be the inverse to f on $f(U)$. The final step consists in showing that $\varphi(w)$ is analytic on $f(U)$. This gives the inverse-function theorem in the small. The entire proof is completely different from the classical one which proceeds very rapidly via the Rouché theorem. The fact that $[f'(z)]^{-1}$ exists and is bounded in a neighborhood of a is heavily used both here and later. As an easy corollary of the open-mapping theorem above the author shows that the range of $f(z)=\exp(z)$ is precisely the component containing the identity of the group of elements regular in B. This result is due to Nagumo [Japan. J. Math. **13** (1936), 61–80].

Next, the author considers the notions of function element, of analytic continuation, and of analytic function in the large. These are introduced in the standard way. The theorem of the invariance of functional relations is established. Then we are ready for the inverse theorem in the large. Let $f(w)$ be analytic in the entire space B. Let $D=\{w: [f'(w)]^{-1}\text{ exists}\}$. Then D is the union of open maximal connected components. Consider one of them, D^*. Then the totality of inverse-function elements of $f(w)$ at the points of D^* defines an analytic function (in

the large) $\Psi(z)$ which is called the inverse function of $f(w)$ with respect to D^*. If we write $H=f(D^*)$, then H is connected and open and $\Psi(z)$ cannot be continued analytically beyond H. Let $p_a(z)$ be a function element (power series) with center a of the inverse function $\Psi(z)$ and let $\rho(p_a(z))$ be the radius of convergence of $p_a(z)$. Then $\rho(p_a(z)) \leq \rho(a, H)$ is easily proved. For further results the author hypothesizes $\rho(p_a(z))=\rho(a, H)$. He does not give examples showing the necessity of this condition. The author applies his inverse-function theorem to the function $f(w)=\exp(w)$, in which case he finds again results given by the reviewer in another form, and to $f(w)=w^m$ where m is a positive integer. *E. R. Lorch* (New York)

9881:

Anosov, V. I. Critical points of periodic functionals. Dokl. Akad. Nauk SSSR **131** (1960), 223–225 (Russian); translated as Soviet Math. Dokl. **1**, 208–210.

Let E be a regular Banach space with uniformly differentiable norm, and let A be a periodic (not necessarily linear) operator that maps the unit sphere $S=\{x: \|x\|=1\}$ onto itself. Results that are generalizations of earlier work of Krasnosel'skiĭ [same Dokl. **103** (1955), 961–964; MR **18**, 406] on critical points and values of functionals invariant under A are announced; the main theorem is as follows. Let F be a weakly continuous, uniformly differentiable real function on S such that (1) $F(x)=F(Ax)$ $(x \in S)$, (2) $F(0)=0$, $F(x)>0$ if $x \neq 0$, (3) $\Gamma(0)=0$, $(\Gamma x, x) \neq 0$ $(x \neq 0)$ where $\Gamma=\text{grad } F$. Assume that A satisfies a Lipschitz condition on S, and call $x_0 \in S$ a critical point of F if $\Gamma x_0=\lambda L_\alpha x_0$ where $\lambda \neq 0$ and $L_\alpha=\text{grad } \|x\|^\alpha$ $(\alpha>1)$. Theorem: F has at least a countable number of distinct critical points on S. The method of proof follows an outline suggested by Krasnosel'skiĭ [*Topologičeskie metody v teorii nelineĭnyh integral'nyh uravneniĭ*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956; MR **20** #3464]. *H. H. Schaefer* (Ann Arbor, Mich.)

9882:

Bremermann, H. J. The envelopes of holomorphy of tube domains in infinite dimensional Banach spaces. Pacific J. Math. **10** (1960), 1149–1153.

A complex-valued functional f defined in a domain D of a complex Banach space B is called G-holomorphic (=Gâteaux-holomorphic), if the restriction of f into the intersection $D \cap L$ of D with any finite-dimensional affine subspace $L=\{z|z=z_0+\tau_1 a_1+\cdots+\tau_n a_n\}$ is holomorphic in the ordinary sense. In this case, there occurs the phenomenon of simultaneous holomorphic continuation, similar to the case of a finite (>1) number of complex variables.

In the present paper, the author proves that the envelope of G-holomorphy of a tube domain $T_X=\{z=x+iy|x \in X, y \text{ arbitrary}\}$ is its convex-hull. For the case of finite-dimensional space B, this is a well-known result. The proof depends on the fundamental lemma concerning the plurisubharmonicity of the function $-\log d_D(z)$ for a domain of G-holomorphy, where $d_D(z)$ is the boundary distance of a point z in the domain D. The proof seems much simpler than the previous proofs for finite dimension. The author also declares that this is the first time that the envelope of holomorphy of an infinite-dimensional domain has been actually determined.

S. Hitotumatu (Tokyo)

CALCULUS OF VARIATIONS

See also 9453, 9701.

9883:

Lombardi, Lionello. Sur l'application de la méthode du gradient au calcul des variations. Les mathématiques de l'ingénieur, pp. 216–225. Mém. Publ. Soc. Sci. Arts Lett. Hainaut, vol. hors série, 1958.

The present paper is concerned with the concept of the gradient of various functionals arising in the calculus of variations. After reviewing the concept and the use of the gradient of functions of a finite number of variables, the author discusses an extension of these results to simple integrals, to multiple integrals and to Fubini-Tonelli integrals. By modifying the inner product that was introduced, one can obtain further extensions of the concept of the gradient of a functional.

M. R. Hestenes (Los Angeles, Calif.)

9884:

Picone, Mauro. Su un criterio sufficiente in un classico problema di calcolo delle variazioni. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **94** (1959/60), 341–352.

In a recent note [Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Nat. (8) **28** (1960), 131–138; MR **22** #5918] the author gave a new sufficient condition that an extremal $z(x, y)$ of a variational integrand $f(x, y, z, z_x, z_y)$ furnish a minimum. The present paper is devoted to special cases and examples which illustrate this condition.

W. H. Fleming (Providence, R.I.)

9885:

Plotnikov, V. I. The semicontinuity of functionals of the calculus of variations. Mat. Sb. (N.S.) **52 (94)** (1960), 799–810. (Russian)

The interest of the rather general semi-continuity theorem established by the author is that it is independent of notions such as that of Jacobian, or generalized Jacobian. Let V, W denote Euclidean spaces of dimensions l and m, let X denote a bounded domain in Euclidean space of dimension k, and let F denote a real continuous function $F(x, v, w)$ on the Cartesian product of X, V, W which is convex in w for each x, v. Further, given a pair of measurable functions $v(x)$, $w(x)$ which map X into V, W, we write I_0 for the integral over X of $F[x, v(x), w(x)]$; similarly I_n $(n=1, 2, \cdots)$ will denote its values when the pair of functions are assigned the suffix n; moreover, these notations will not be regarded as affected when $v(x)$, $w(x)$ are defined almost everywhere, instead of everywhere, in X. The author's theorem then asserts that $\liminf I_n \geq I_0$ whenever it is supposed that $v_n(x)$ tends in measure to $v(x)$ and for every closed set $F \subset X$ the integrals over F of $w_n(x)$ tend to that of $w(x)$, provided that in addition (i) the absolute values $\|w_n(x)\|$ have in X uniformly bounded integrals and (ii) F is non-negative and possesses continuous partial derivatives in the components of w. {Reviewer's remark: It almost goes without saying that (ii) can be relaxed.} *L. C. Young* (Madison, Wis.)

9886:

Šilova, G. I. Existence of absolute minimum of multiple integrals in calculus of variations in non-parametric form. Mat. Sb. (N.S.) **51 (93)** (1960), 253–272. (Russian)

A proof is given of an existence theorem announced

previously [Dokl. Akad. Nauk SSSR **102** (1955), 699–702; MR **17**, 46] for the non-parametric problem in n independent variables in the so-called borderline case $\alpha=n$, α being the "order" of the variational integrand $F(x^1, \cdots, x^n, z, p^1, \cdots, p^n)$. The author makes extensive use of machinery developed by A. G. Sigalov for the 2-dimensional case [Trudy Moskov. Mat. Obšč. **2** (1953), 201–233; Uspehi Mat. Nauk (N.S.) **6** (1951), no. 2 (42), 16–101; MR **15**, 442; **13**, 257]. An additional tool is an estimate of the oscillation of a function of class A^α on the boundary of a parallelepiped, based on the isoperimetric inequality. *W. H. Fleming* (Providence, R.I.)

9887:

Davies, E. T. Applicazioni del calcolo delle variazioni alla geometria differenziale. Confer. Sem. Mat. Univ. Bari **47**, 11 pp. (1959).

This is an excellent survey on some modern results in the geometry of the calculus of variations.

E. Bompiani (Rome)

GEOMETRY

See also 9447, 9453.

9888:

Hohenberg, Fritz. ★Konstruktive Geometrie in der Technik. 2te, neubearbeitete und erweiterte Aufl. Springer-Verlag, Vienna, 1961. ix+319 pp. $6.90.

Die 1. Auflage erschien 1956 als *Konstruktive Geometrie fur Techniker* und wurde hier [MR **17**, 1234] sehr lobend besprochen. In den Hauptsachen ist es ungeändert, aber an zahlreichen Stellen ist die Darstellung ausführlicher (die erste Auflage hatte 272 Seiten). Neu hinzugekommen sind der Robertsche Satz, die Wattschen Kurven und die geometrischen Grundlagen des Wankel-Motors.

O. Bottema (Delft)

9889:

Thébault, Victor. Géométrie du polygone gauche à sommets cosphériques. Mathesis **68** (1959), 243–249.

On considère un polygone gauche P dont les sommets A_i sont sur une sphére. Les normales menées des sommets à un plan formé de trois sommets déterminent sur la sphère circonscrite un second polygone P'; relations liant les distances des barycentres et centres des sphères de Kantor de P et P' aux distances des sommets de P et P' au plan; l'auteur en déduit comme cas particuliers des propriétés connues. Lieu du centre des sphères Q de rayon r, dont les puissances des sommets $A_iQ^2-r^2=mK_i^2$ sont proportionnelles aux sommes des carrés des distances des sommets des polygones P_i formés des sommets de P autres que A_i; cas particuliers. Etude de quelques sphères liées au polygone P, en particulier des sphères fondamentales associées aux polygones p_i formés des $n-1$ sommets autres que A_i et du point de Kantor de P pris dans un certain ordre. *B. d'Orgeval* (Dijon)

9890:

Marmion, A. Tétraèdres et sphères podaires relativement à 4 droites de l'espace. Mathesis **69** (1960), 156–197.

Section I treats the case of four arbitrary lines. Given in space four straight lines D_i and a point P, the pedal tetrahedron of P for the lines is formed by the projections P_i onto the lines; the circumscribed sphere of this tetrahedron is the pedal sphere of P. The author gives the properties of two points P and P' having the same pedal sphere for D_i, and finds that the pedal sphere of a line Δ for four lines D_i is orthogonal to a sphere Ω, orthopedal of Δ. Then the properties of the envelope surface N', which is of the 6th class, of the pedal planes for four lines are discussed. The interesting study establishes 51 propositions. It concludes with the properties of the pedal spheres for 5 and 6 lines.

In section II the results are treated when the four lines are hyperboloidic. Section III is devoted to the case when the four lines are the sides of a skew quadrilateral. Section IV studies the case when the four lines are parallel to a horizontal plane.

D. Mazkewitsch (Cincinnati, Ohio)

9891:

Pforr, Ernst-Adam. Einfache geometrische Räume. Math. Nachr. **20** (1959), 265–301.

The purpose of this paper, part of a dissertation, is the characterization of the affine, projective and higher geometries by means of manifolds, in analogy to their use in higher differential geometry [cf. O. Veblen and J. H. C. Whitehead, *The foundations of differential geometry*, Cambridge Univ. Press, London, 1932]. The author introduces "geometric $\mathfrak{R}^n$-spaces" which are generalized manifolds; then he considers special types of these spaces, "simple geometric spaces".

Let M be a non-empty set, $\mathfrak{R}$ a ring with unit element, $\mathfrak{R}^n$ an n-dimensional vector space over $\mathfrak{R}$, $\mathfrak{P}(\mathfrak{R}^n)$ the power set of $\mathfrak{R}^n$. Then a "general $\mathfrak{R}^n$-coordinate system" for M is defined as a biunique mapping κ of $A\subset M$ into $\mathfrak{P}(\mathfrak{R}^n)$ with the property that $x\neq y$ implies $\kappa(x)\cap\kappa(y)=\emptyset$. Furthermore, let $\mathfrak{K}$ be a system of general $\mathfrak{R}^n$-coordinate systems for M such that for each $x\in M$ there is a $\kappa\in\mathfrak{K}$ which has x in its domain. Then the couple $(M,\mathfrak{K})$ is called a "geometric $\mathfrak{R}^n$-space" if there exists a pseudo-group [Veblen and Whitehead, op. cit.] $\mathfrak{P}$ of $\mathfrak{P}(\mathfrak{R}^n)$ into itself satisfying the following requirements: (1) If for two coordinate systems κ, κ' from $\mathfrak{K}$, $\kappa'\kappa^{-1}$ exists, then $\kappa'\kappa^{-1}\in\mathfrak{P}$. (2) $\kappa\in\mathfrak{K}$, $g\in\mathfrak{P}$ and the existence of $g\cdot\kappa$ imply $g\cdot\kappa\in\mathfrak{K}$. (3) The union of arbitrarily many coordinate systems from $\mathfrak{K}$, if it exists, belongs again to $\mathfrak{K}$. A geometric $\mathfrak{R}^n$-space $(M,\mathfrak{K})$ is called "simple" if there is a $Z_0\subset\mathfrak{P}(\mathfrak{R}^n)$ such that for each $\kappa\in\mathfrak{K}$, M is the domain of κ, and Z_0 its range.

The author proves that all spaces of affine and projective geometry, Plücker's line-space, Lie's sphere-space and the space of oriented lines are simple geometric spaces, and he admits that a much narrower definition of simple geometric spaces would have been sufficient for these applications. The paper also includes investigations of the relations between the different spaces.

{The reviewer has doubts concerning the following somewhat overly ambitious statement in the introduction of the paper where the author proposes to define the task of Higher Geometry as the investigation of "simple geometric spaces": "Allerdings hat man sich unseres Wissens bisher keine Gedanken über den Geltungsbereich der 'höheren Geometrie' gemacht, obwohl man von 'höherer Geometrie' spricht und auch Vorlesungen über 'höhere Geometrie' zu halten pflegt."}

R. Artzy (Chapel Hill, N.C.)

9892:

Court, N. A. Four intersecting spheres. Amer. Math. Monthly **67** (1960), 241–248.

Author's summary: "Four intersecting spheres (A), (B), (C), (D), taken three at a time, determine four pairs of points (ε). To study the configuration formed by those points we consider: (a) the sphere (M) orthogonal to the given spheres; (b) the tetrahedron $(T)=ABCD$ formed by the centers of those spheres; (c) the polar reciprocal tetrahedron $(T')=A'B'C'D'$ of (T) with respect to (M); (d) the 'adjoint spheres' (A'), (B'), (C'), (D'), with centers at the vertices of (T') and orthogonal to (M). The eight points (ε) give rise to $2^4=16$ 'tetrahedrons of intersection' of the given spheres. They may be grouped into eight 'pairs of complementary tetrahedrons of intersection' (abbreviated to c.t.i.), each pair involving all the eight points. The tetrahedrons of each pair are perspective, the center of perspectivity being the center M of the sphere (M). Their eight planes of perspectivity coincide with the eight planes of similitude of the four adjoint spheres. The radical plane of the circumspheres of a pair of c.t.i. coincides with the plane of perspectivity of the two tetrahedrons considered. The two spheres are inverse with respect to (M), are cut isogonally by each of the given spheres, etc." *S. R. Struik* (Cambridge, Mass.)

9893:

Hecquet, J. Sur les sphères adjointes à un tétraèdre. Mathesis **69** (1960), 37–44.

Un tétraèdre T possède 24 cercles adjoints (cercles passant par 2 des 3 sommets et par l'un des 2 points de Brocard d'un triangle). A chaque cercle adjoint de T on ne peut faire correspondre que 3 autres cercles adjoints de façon à définir une sphère, appelée par analogie: sphère adjointe. Leur nombre est de 36. Ces 36 sphères peuvent être groupées en 3 familles de 12 sphères de la façon suivante: (a) sphères passant par 3 sommets de T et tangentes à une arête en l'un de ces sommets; (b) sphères passant par 2 sommets de T et tangentes à une face en l'un de ces sommets; (c) sphères tangentes à 2 arêtes opposées de T et passant par 2 sommets de T.

L'auteur étudie dans chacun de ces cas certaines propriétés géométriques qui en découlent.

F. Şemin (Istanbul)

9894:

Barrucand, P. Sur la cubique de Thomson et les foyers de Steiner. Mathesis **69** (1960), 44–48.

Author's introduction: "Nous avons établi [Mathesis **66** (1957), 69–70] que dans un triangle ABC deux points réciproques quelconques P', Q' et leurs complémentaires P, Q appartiennent à une conique η circonscrite à ABC [J. Bilo, ibid. 324–325]. La présente note considère le cas où P', Q' se trouvent sur la cubique de Lucas, ce qui place P, Q sur la cubique de Thomson, pour supposer ensuite P' en un foyer de l'ellipse de Steiner circonscrite au triangle." *A. W. Goodman* (Lexington, Ky.)

9895:

Bilo, J. Quadruples hyperboloïdiques. Mathesis **68** (1959), 249–253.

(A) étant un tétraèdre de sommets A_i et de faces opposées α_i, Σ une quadrique tangente en P_i aux faces α_i, R. Deaux et J. Depunt ont montré [Mathesis **68** (1959), 7–13; MR **21** #7473] que pour que les droites A_iP_i concourent il faut et il suffit que les génératrices de Σ situées dans l'une des α_i soient tangentes à la conique touchant les arêtes qu'elle contient aux pieds des céviennes de P_i. L'auteur envisage le cas général où les A_iP_i forment un quadruple hyperboloïdique. Etant donnés deux tels quadruples, A_iP_i et A_iQ_i portés par une même quadrique Π, et les deux quadriques Σ_p, Σ_q inscrites dans (A) tangentes à α_i en P_i et Q_i, dont les génératrices situées dans α_i sont respectivement (p_i, p_i'), (q_i, q_i'), il montre que (p_i, p_i') sont les tangentes issues de P_i à la conique touchant les arêtes de (A) situées dans α_i aux pieds des céviennes de Q_i, la corde des contacts de ces tangentes étant alors l'intersection de α_i avec le plan $(P_jP_kP_l)$ polaire de A_i par rapport à Σ_p. Il montre aussi que p_i et p_i' sont conjuguées dans l'inversion tangentielle J attachée à la face α_i, que les points d'intersection de (p_i, p_i') avec (q_i, q_i'), $i=1, 2, 3, 4$, et les sommets de (A) sont 20 points d'une même quadrique, et que les quadriques Σ_p, Σ_q (considérées comme quadriques-enveloppes) sont invariantes dans l'inversion tangentielle tétraèdrique J. Il termine par un critère pour que les droites A_iP_i relatives à une Σ_p tangente aux faces d'un tétraèdre (A) forment un quadruple hyperboloïdique, ou bien deux couples plans, ou bien un système de 4 droites concourantes. *P. Vincensini* (Marseille)

9896:

Whitehead, A. N. ★The axioms of descriptive geometry. Reprinting of Cambridge Tracts in Mathematics and Mathematical Physics, No. 5. Hafner Publishing Co., New York, 1960. viii+74 pp. $3.00.

A reprint of the 1907 edition [Cambridge Univ. Press, London].

9897:

Zawadzki, Adam. ★Trójobrazowy rzut równoległy [Three-image parallel projection]. Państwowe Wydawnictwo Naukowe, Warsaw, 1957. 148 pp. zł 19.50.

This book contains the generalisation of Monge's projections of the three planes of an oblique coordinate system where the axis is the direction of projection on the plane of the other two axes. The author considers the basic graphic and metric constructions and he applies them to spherical trigonometry. *E. Otto* (Warsaw)

9898:

Havel, Václav. Zur Geometrie der Translationsebenen. Czechoslovak Math. J. **10** (85) (1960), 432–439. (Russian summary)

Given an affine plane satisfying the Fano configuration, the author shows that if any one of five subsidiary configuration theorems is satisfied, the plane is a translation plane. The author also gives a set of configuration theorems such that a projective plane satisfying the Fano configuration and some of these conditions is an alternative plane. The entire work is carried out synthetically.

N. S. Mendelsohn (Winnipeg, Man.)

9899:

Wood, P. W. ★The twisted cubic, with some account of the metrical properties of the cubical hyperbola. Reprinting of Cambridge Tracts in Mathematics and Mathematical Physics, No. 14. Hafner Publishing Co., New York, 1960. x+78 pp. (1 insert) $3.00.

A reprint of the 1913 edition [Cambridge Univ. Press, London].

9900:

Jacob, M. Les coordonnées polaires tangentielles et leurs applications. Les mathématiques de l'ingénieur, pp. 211–215. Mém. Publ. Soc. Sci. Arts Lett. Hainaut, vol. hors série, 1958.

Starting from a plane polar coordinate system with pole O and points $P=(p, \alpha)$, a new system of line-coordinates is introduced in which (p, α) is the straight line through P, perpendicular to OP. If in the original polar coordinate system $p=f(\alpha)$ is a curve, then in the line-coordinate system the same equation represents the anti-pedal of this curve. Properties, examples, applications to mechanics and a 3-dimensional generalization are mentioned.

R. Artzy (Chapel Hill, N.C.)

9901:

Marchionna, Ermanno. Sulle superficie aritmeticamente regolari ed aritmeticamente normali. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **24** (1958), 24–35.

Sia F una superficie algebrica di generi p_g, p_a appartenente allo spazio S_r. Si dica F "aritmeticamente regolare" se le forme d'ordine l qualsiasi segano su F un sistema lineare regolare; "topologicamente regolare" se $q=p_g-p_a=0$; "aritmeticamente normale" se le forme d'ordine l qualsiasi segano su F un sistema lineare completo. Ciò posto, siano Ψ, Ψ' due superficie di S_r non singolari, di irregolarità (topologiche) q, q', e complementari, formanti insieme la completa intersezione di $r-2$ ipersuperficie d'ordini $n_1, n_2, \cdots, n_{r-2}$; siano Δ_l ed s_l la deficienza e la sovrabbondanza del sistema segato su Ψ dalle ipersuperficie d'ordine l di S_r ($\Delta_l=0$ per ogni l se Ψ è aritmeticamente normale, $s_l=0$ per ogni l se Ψ è aritmeticamente regolare); e Δ_l', s_l' i caratteri analoghi di Ψ'. L'autore dimostra dapprima che se $l>\rho$ ($\rho=n_1+n_2+\cdots+n_{r-2}-r-1$) si ha: $s_l=s_l'=\Delta_l=\Delta_l'=0$; se $l=\rho$ si ha $s_\rho=s_\rho'=0$, $\Delta_\rho=q'$, $\Delta_\rho'=q$; se $l<\rho$ è: $\Delta_l+\Delta'_{\rho-l}=s_l+s'_{\rho-l}$. Da questo risultato segue immediatamente, poichè Δ_l, s_l, Δ_l', s_l' son tutti non negativi, che condizione necessaria e sufficiente affinchè due superficie complementari, non singolari, siano insieme aritmeticamente normali ($\Delta_l=\Delta_l'=0$, per ogni l) è che esse siano topologicamente ed aritmeticamente regolari ($q=q'=s_l=s_l'=0$, per ogni l). Un esempio di due superficie complementari (e non singolari) una delle quali sia aritmeticamente normale e l'altra no era già stato indicato dall'autore [#9902]; circostanza questa che non ha l'analoga nella teoria delle curve.

L'autore dimostra inoltre che la generica sezione iperpiana di una superficie Ψ aritmeticamente normale (non singolare) risulta essa pure aritmeticamente normale se e solo se una superficie Ψ' (non singolare) complementare di Ψ è aritmeticamente normale.

D. Gallarati (Genoa)

9902:

Marchionna, Ermanno. Sulle varietà di prima specie. Univ. e Politec. Torino. Rend. Sem. Mat. **16** (1956–57), 433–438.

Si on considère dans l'espace projectif complexe S_r une variété algébrique V_d à d dimensions, et une de ses sections fixée $\overline{V}_{d-1}$ par un hyperplan de S_r, on dit V_d de première espèce si les hypersurfaces d'ordre l quelconque passant par V_d coupent sur l'hyperplan un système complet d'hypersurfaces d'ordre l contenant $\overline{V}_{d-1}$; on dit V_d arithmétiquement normale si les hypersurfaces d'ordre l coupent sur V_d un système linéaire complet: ces deux propriétés sont équivalentes lorsque V_d est irréductible sans points multiples. Lorsque l'intersection des hypersurfaces par V_d contient une autre partie, l'on sait qu'au cas de courbes sans points multiples elles sont de même espèce; cette propriété ne se maintient pas si $d>1$. L'auteur construit un contre-exemple: on considère dans S_4 deux variétés cubiques passant par une surface quartique F projection de la surface de Veronese et par une surface résiduelle du 5° ordre, toutes deux sans singularités; la quartique n'est pas normale au sens ordinaire, donc non arithmétiquement normale, et la quintique l'est comme on le voit en étudiant ses sections hyperplanes génériques; sur la quintique les sections ne sont pas non plus arithmétiquement normales, ce qui contredit l'inverse: à savoir, si les sections sont arithmétiquement normales, la surface l'est aussi. L'auteur montre que l'on peut former des chaînes de surfaces d'espèces distinctes, et en calcule les irrégularités.

B. d'Orgeval (Dijon)

9903:

Nevin, Jennifer M. On Cremona transformations without fundamental points. Rend. Mat. e Appl. (5) **19** (1960), 122–123.

Semplice dimostrazione del teorema secondo cui una trasformazione cremoniana tra due spazi proiettivi n-dimensionali la quale in uno dei due spazi non ha punti fondamentali, è una trasformazione proiettiva.

D. Gallarati (Genoa)

9904:

Buquet, A. Sur les points rationnels d'une biquadratique gauche et d'une quartique binodale. Mathesis **69** (1960), 143–156.

The author devotes his first ten pages to explaining an additive calculus of point sets on the elliptic quartic, which is in fact nothing but the familiar one of coresiduation modulo plane sections of the twisted curve, or modulo sections of the plane curve by adjoint conics (the additive calculus of the elliptic parameters modulo periods). He then points out that if the tetrahedron of reference has vertices I, J, K, L on the twisted quartic, where $J\equiv -3I$, $K\equiv 5I$, $L\equiv -7I$, two quadrics through the curve have the form

$$ayz+czy+eyt+fzt=0, \quad bxz+dxt+gyt=0.$$

In terms of these seven coefficients he obtains the coordinates of the points $9I$, $-11I$, $13I$, $-15I$, $17I$, $\cdots$. The point $9I$ is coplanar with IKL. If $e=0$, $16I\equiv 0$, so that I coincides with $-15I$ and with $17I$; and similarly the conditions for $20I\equiv 0$, $24I\equiv 0$ are $ade-cfg=0$, $ade-cfg-bc^2=0$.

P. Du Val (London)

9905:

Henderson, Archibald. ★The twenty-seven lines upon the cubic surface. Reprinting of Cambridge Tracts in Mathematics and Mathematical Physics, No. 13. Hafner Publishing Co., New York, 1960. vii+100 pp. (14 inserts) $3.00.

A reprint of the 1911 edition [Cambridge Univ. Press, London].

9906:

Gallarati, Dionisio. Sulle irregolarità di un S_r doppio. Boll. Un. Mat. Ital. (3) **14** (1959), 175–181. (English summary)

Estendendo alcuni risultati ottenuti dall'A. sugli S_3 doppi irregolari, si determina qui il numero delle forme differenziali olomorfe dei vari gradi di una varietà algebrica V_r, superficialmente irregolare, che sia rappresentabile doppiamente su uno spazio proiettivo S_r, con una varietà di diramazione costituita da 2μ ($\mu \geq 2$) ipersuperficie di ordine b, appartenenti ad un fascio generale. Detti g_i ($i=1, \cdots, r$) i generi geometrici di V_r ($g_i = \dim_C H^i(V_r, \Omega)$, Ω fascio dei germi delle funzioni olomorfe), il risultato è il seguente:

$$g_r = \mu\binom{b-1}{r}, \quad g_k = 0 \quad (1 < k < r), \quad g_1 = \mu - 1.$$

F. Gherardelli (Florence)

9907:

Whitehead, A. N. ★The axioms of projective geometry. Reprinting of Cambridge Tracts in Mathematics and Mathematical Physics, No. 4. Hafner Publishing Co., New York, 1960. viii+64 pp. $3.00.

A reprint of the 1906 edition [Cambridge Univ. Press, London].

9908:

Lotze, Alfred. Über die von Seydewitzsche Erzeugung von Quadriken und ihre m-dimensionale Verallgemeinerung. Jber. Deutsch. Math. Verein. **63**, Abt. 1, 85–88 (1960).

9909:

Marcus, F. L'élément linéaire projectif d'une congruence de droites dans S_5. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. Mat. **10** (1959), 129–140. (Romanian. Russian and French summaries)

L'auteur introduit l'élément linéaire projectif d'une congruence de droites dans S_5; pour les congruences quadratiques il coïncide avec celui de B. Segre [Acad. Roy. Belg. Bull. Cl. Sci. (5) **39** (1953), 481–489; MR **15**, 156]. On généralise la notion de l'applicabilité projective de la deuxième espèce [Terracini, Ist. Veneto Sci. Lett. Arti Atti Cl. Sci. Mat. Nat. **94** (1945), 75–86].

A. Švec (Prague)

9910:

Terracini, Alessandro. Sull'elemento lineare proiettivo di una congruenza di rette nello spazio a cinque dimensioni. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **28** (1960), 3–7.

L'auteur étudie les relations qui existent entre les éléments linéaires projectifs d'une congruence de droites d'S_5 introduits par F. Marcus [#9909] et A. Švec [Czechoslovak Math. J. **9** (**84**) (1959), 243–264; MR **21** #5969] et donne leur interprétation géométrique à l'aide de ses recherches antérieures [Un. Mat. Argentina Publ. No. 16 (1940); MR **2**, 300].

A. Švec (Prague)

9911:

Mandan, Sahib Ram. Cevian simplexes. Proc. Amer. Math. Soc. **11** (1960), 837–845.

Let $A_1A_2 \cdots A_{n+1}$ be a simplex in projective n-space, a^i the $(n-1)$-space opposite to the vertex A_i, a^{ij} the $(n-2)$-space opposite to the edge A_iA_j, M an arbitrary point, A_i' the point $a^i \cdot A_iM$, and M_{ij} the point $Ma^{ij} \cdot A_iA_j$. The author observes that the "Cevian" simplex $A_1'A_2' \cdots A_{n+1}'$ is perspective with the original simplex $A_1A_2 \cdots A_{n+1}$ from the point M and from its harmonic polar hyperplane m ("trilinear polar" when $n=2$), that the two simplexes are related by a homology with center M and axial hyperplane m, and that the cross-ratio of this homology is $-n$. He proves many theorems, among which the following is typical: There exists a quadric touching every edge A_iA_j at M_{ij}, the tangent hyperplane at this point being $(m \cdot a^{ij})A_iA_j$ [cf. A. Hameed, Pakistan J. Sci. Res. **3** (1951), 48–51; MR **14**, 494].

H. S. M. Coxeter (Toronto)

9912:

Hughes, D. R. On homomorphisms of projective planes. Proc. Sympos. Appl. Math., Vol. 10, pp. 45–52. American Mathematical Society, Providence, R.I., 1960.

A homomorphism of a projective plane π onto a projective plane π_1 is an incidence-preserving map carrying the points of π onto the points of π_1 and the lines of π onto the lines of π_1. The author introduces coördinates in a given plane, using the construction he described in Trans. Amer. Math. Soc. **80** (1955), 502–527 [MR **17**, 451]. The "coördinate-system" consists of elements from a set R on which is defined a ternary function F. Each line $[m, k]$ not through Y (the ideal point on $\{x=0\}$) consists of an ideal point together with the ordinary points (x, y) such that $F(m, x, y)=k$. The function F satisfies: (A) $F(a, 0, c)=F(0, b, c)=c$ for all $a, b, c \in R$. (B) $F(a, 1, 0)=F(1, a, 0)=a$ for all $a \in R$. (C) If $a, b, c, d \in R$, $a \neq c$, then there is a unique $x \in R$ such that $F(x, a, b)=F(x, c, d)$. (D) If $a, b, c, d \in R$, $a \neq c$, there is a unique ordered pair $x, y \in R$ such that $F(a, x, y)=b$, $F(c, x, y)=d$. (E) If $a, b, c \in R$, there is a unique $x \in R$ such that $F(a, b, x)=c$. If R is non-empty and F is a ternary function on R, then (R, F) is called a ternary ring. If (R, F) is a ternary ring with at least two distinct elements 0, 1, and satisfying (A)–(E), then (R, F) is called a planar ternary ring. Addition and multiplication are defined in R by $a+b=F(1, a, b)$ and $a \cdot b=F(a, b, 0)$. If $F(a, b, c)=ab+c$ for all $a, b, c \in R$, the ternary ring is called linear.

The author proves the following results. (I) If (R, F) is a planar ternary ring and S is a finite subset of R such that (S, F) is a ternary ring, then either S consists of zero alone, or (S, F) is a planar ternary ring. (II) If φ is a homomorphism of π onto π_1 mapping $0:(0, 0)$ onto $0_1:(0, 0)$ and the ideal points $Y:(\infty)$, $X:(0)$, $U:(1)$ of π onto the respective ideal points $Y_1:(\infty)$, $X_1:(0)$, $U_1:(1)$ of π_1, then φ is an algebraic homomorphism in the sense that $[F(x, y, z)]\varphi = F_1(x\varphi, y\varphi, z\varphi)$ for all $x, y, z \in R_0 = R-J$, with J the set of all elements y for which $(0, y)\varphi =$

$Y_1:(\infty)$. [Note: Given any homomorphism of π onto π_1 and the points 0_1, Y_1, X_1, U_1 of π_1, points 0, Y, X, U can then be chosen in π to satisfy the hypotheses of (II).] (III) If Z is the set of all y such that $(0, y)\varphi=(0, 0)$, then (R_0, F) and (Z, F) are ternary rings; addition in (R_0, F) forms a loop, under which $(Z, +)$ is a normal subloop; for $x, y \in R_0$, $x\varphi=y\varphi$ if and only if $x=z+y$, with $z \in Z$. (IV) If $(M, \cdot)$ is a loop and if $I=Z+1$, then $(I, \cdot)$ is a normal subloop of $(M, \cdot)$. (V) Since (Z, F) is never planar, Z is infinite or $Z=0$. Corollaries include: (1) If φ is a homomorphism of π onto π_1 and if φ is not an isomorphism, the number of points on a given line L of π which map onto a single point of $L\varphi$ is infinite. (2) A finite projective plane possesses no homomorphisms except isomorphisms. The author remarks that converses of (I)–(V) can be stated but are complicated to derive.

The author concludes with three theorems which describe how collineations on π may be "preserved" under a homomorphism. (VI) Let φ denote a collineation of π onto π_1, and θ, a central collineation on π with axis K and center Q ($Q \in K$). If there is a point P_1 of π such that both $P_1\varphi$ and $(P_1\theta)\varphi$ fail to lie on $K\varphi$, then the map $\theta_1: P\varphi \to (P\theta)\varphi$, $L\varphi \to (L\theta)\varphi$, for points $P\varphi$ of π_1 and lines $L\varphi$ of π_1, is a central collineation of π_1 with center $Q\varphi$, axis $K\varphi$. (VII) Let φ denote a homomorphism of π onto π_1. Let θ be a central collineation of π with axis K and center Q, where Q is not on K, and also $Q\varphi$ is not on $K\varphi$. If there is a point P_1 of π such that $P_1\varphi$ and $(P_1\theta)\varphi$ are both not on $K\varphi$, and neither equals $Q\varphi$, then the mapping $\theta_1: P\varphi \to (P\theta)\varphi$, $L\varphi \to (L\theta)\varphi$, for points $P\varphi$ of π_1 and lines $L\varphi$ of π_1, is a central collineation of π_1 with axis $K\varphi$ and center $Q\varphi$. (VIII) Let φ denote a homomorphism of π onto π_1. Let θ be a central collineation of π with axis K and center Q, where Q is not on K but $Q\varphi$ is on $K\varphi$. If there is a point P_1 of π such that $P_1\varphi$ and $(P_1\theta)\varphi$ both fail to lie on $K\varphi$, then the mapping $\theta_1: P\varphi \to (P\theta)\varphi$, $L\varphi \to (L\theta)\varphi$, for points $P\varphi$ of π_1 and lines $L\varphi$ of π_1, is a central collineation of π_1 with axis $K\varphi$ and center $Q\varphi$.

The author feels that homomorphisms may help to answer some outstanding questions about finite projective planes, any such plane being the homomorphic image of a free plane. Specifically, light may be shed on the problem of whether every finite plane has non-trivial collineations.

{In line 13, page 46, $(0, k)$ is a misprint and should read $(k, 0)$.} *W. A. Pierce* (Syracuse, N.Y.)

9913:

Blaschke, Wilhelm. Sulle congruenze isotrope nello spazio ellittico. Ann. Mat. Pura Appl. (4) **50** (1960), 115–117.

Let Q be the absolute quadric of an elliptic 3-dimensional space E_3; in an appropriate coordinate system let the equation of Q be: $\sum_i x_i^2=0$ ($i=0, \cdots, 3$), or in parametric form $x_0=i(1+zw)$, $x_1=z+w$, $x_2=-i(z-w)$, $x_3=1-zw$, where $z=\text{const}$, $w=\text{const}$ represent the generators of Q.

The point

$$Z\left(\frac{z+\bar{z}}{1+z\bar{z}}, \frac{z-\bar{z}}{i(1+z\bar{z})}, \frac{1-z\bar{z}}{1+z\bar{z}}\right)$$

($\bar{z}$ complex conjugate of z) describes the Riemann sphere of z; similarly for W (changing z to w). The group G_6 of elliptic motions in E_3 is the free product of the rotations of the two spheres. A real oriented line of E_3 is determined by its complex conjugate points of intersection with Q, i.e., by a pair of points on the two spheres. A conformal mapping of the two spheres, $w=f(z)$ with f analytic, determines a congruence of lines; their polars with respect to Q determine an isotropic congruence of lines, analogous to those considered by Ribaucour in a Euclidean space. To find a possible extension to the elliptic space of the links existing in the Euclidean space between isotropic congruences and minimal surfaces, the author suggests the study of the group G_{12}, the free product of the group of projectivities of the two Riemannian spheres. *E. Bompiani* (Rome)

CONVEX SETS AND GEOMETRIC INEQUALITIES

9914:

Fulton, Curtis M.; Stein, Sherman K. Parallelograms inscribed in convex curves. Amer. Math. Monthly **67** (1960), 257–258.

The authors show that any convex curve enclosing an area A is such that a parallelogram can be inscribed in the curve of area at least $A/2$ and that equality occurs if and only if the circumscribed curve is a triangle.

H. G. Eggleston (London)

9915:

Grünbaum, Branko. A variant of Helly's theorem. Proc. Amer. Math. Soc. **11** (1960), 517–522.

The following theorem of Helly's type, but for non-convex sets, is proved. Let $K \subset E^k$ be the union of a finite number of disjoint, compact, closed sets. There exists an integer $n=n(K)$ with the following property: For any family $\mathcal{K}=\{K_\alpha: \alpha \in A\}$ of affine transforms of K such that any n members of $\mathcal{K}$ have a nonempty intersection, the intersection $\bigcap_{\alpha \in A} K_\alpha$ is nonempty as well. Further generalizations are also given.

B. Sz.-Nagy (Szeged)

9916:

Chisini, O. Isoperimetri. Period. Mat. (4) **38** (1960), 20–43.

Elementary considerations on the plane isoperimetric problem. An n-angle with given sides has maximum area if it is the (unique) inscribed one. *O. Bottema* (Delft)

9917:

Burlak, Jane A. C.; Rankin, R. A.; Robertson, A. P. The packing of spheres in the space l_p. Proc. Glasgow Math. Assoc. **4**, 22–25 (1958).

The authors make the following extension of Rankin's results in same Proc. **2** (1955), 145–146 [MR **17**, 523]. In the Banach space l_p, the distance between two points $(x_1, x_2, \cdots)$ and $(y_1, y_2, \cdots)$ is $(|x_1-y_1|^p+|x_2-y_2|^p+\cdots)^{1/p}$. Non-overlapping spheres are defined in the natural manner. The question is: How many non-overlapping spheres of fixed radius a can be packed in a sphere of radius 1? The notation $\lambda_p=(1+2^{1-1/p})^{-1}$, $\mu_p=(1+2^{1/p})^{-1}$, $L_1(a)=1$, $L_p(a)=\{1-\frac{1}{2}(a^{-1}-1)^{p/(p-1)}\}^{-1}$, $M_p(a)=\{1-\frac{1}{2}(a^{-1}-1)^p\}^{-1}$ is introduced. If $a \leq \lambda_p$, the

number of spheres may be infinite. If $1 \leqq p \leqq 2$ and $\lambda_p < a \leqq 1$, the maximum number is $L_p(a)$. If $p > 2$ and $\lambda_p < a \leqq \mu_p$ any finite number of spheres can be packed in, though an infinite number cannot. Finally, if $p > 2$ and $\mu_p < a \leqq 1$, the maximum number is $M_p(a)$.

H. S. M. Coxeter (Toronto)

9918:

Molnár, J. Unterdeckung und Überdeckung der Ebene durch Kreise. Ann. Univ. Sci. Budapest. Eötvös. Sect. Math. **2** (1959), 33–40.

In this beautifully illustrated paper, the author continues the work on packing and covering that he began jointly with L. Fejes Tóth [Math. Nachr. **18** (1958), 235–243; MR **20** #2669]. The plane is packed or covered with circles of two radii, $q < 1$ and 1. The problem is to find functions $\delta(q)$, $s(q)$, $S(q)$, $\Lambda(q)$, with $\delta(q) \leqq s(q) < 1 < S(q) \leqq \Lambda(q)$, such that the maximum density of a packing lies between $\delta(q)$ and $s(q)$, and the minimum density of a covering lies between $S(q)$ and $\Lambda(q)$. Since the maximum packing density and minimum covering density are known precisely when $q = 1$, we have $\delta(1) = s(1) = \pi/2\sqrt{3} = 0.9069\cdots$, $S(1) = \Lambda(1) = 2\pi/3\sqrt{3} = 1.2091\cdots$. The functions $s(q)$ and $S(q)$ are taken over from the joint paper already mentioned, where, however, the formulae on p. 243 contain serious errors, e.g., they would make $s(1/2) = \pi/2$, which is greater than 1. The former should be

$$s(q) = \frac{\pi q^2 + 2(1-q^2)\arcsin\{q/(1+q)\}}{2q\sqrt{(1+2q)}}.$$

By examining special arrangements, the author succeeds in finding, for particular values of q, lower bounds $\delta(q)$ that are remarkably close to $s(q)$, and upper bounds $\Lambda(q)$ that are remarkably close to $S(q)$; e.g., $s(0.637) - \delta(0.637) = 0.0004\cdots$, $s(0.533) - \delta(0.533) = 0.0001\cdots$, $\Lambda(1/2) - S(1/2) < 10^{-5}$, $\Lambda(1/4) - S(1/4) < 10^{-5}$.

H. S. M. Coxeter (Toronto)

DIFFERENTIAL GEOMETRY

See also 9546, 9597.

9919:

Libermann, Paulette. Sur les automorphismes infinitésimaux des structures symplectiques et des structures de contact. Colloque Géom. Diff. Globale (Bruxelles, 1958), pp. 37–59. Centre Belge Rech. Math., Louvain, 1959.

In the first part of this article, the author proves the results concerning infinitesimal automorphisms of symplectic structures which were announced in C. R. Acad. Sci. Paris **242** (1956), 1114–1117 [MR **19**, 452]. In the second part, infinitesimal automorphisms of contact structures are studied. In the last part, various examples of cosymplectic and almost cosymplectic structures are discussed.

It is known that a contact structure on an oriented $(2n+1)$-dimensional differentiable manifold V_{2n+1} can be defined by $\omega = 0$, where ω is a 1-form on V_{2n+1} such that $\omega \wedge (d\omega)^n \neq 0$ [cf. John W. Gray, Ann. of Math. (2) **69** (1959), 421–450; MR **22** #3016]. Here ω itself is said to define a Pfaffian structure on V_{2n+1}. Thus a Pfaffian structure is a $1 \times \mathrm{Sp}(n, R)$-structure (i.e., a G-structure in the sense of S.-S. Chern [Colloques Internat. C.N.R.S., Strasbourg, 1953, pp. 119–136, C.N.R.S., Paris, 1953; MR **16**, 112] for $G = 1 \times \mathrm{Sp}(n, R)$), while a contact structure is a $1 \times \mathrm{U}(n)$-structure. In general, any $1 \times \mathrm{Sp}(n, R)$-structure is called an almost cosymplectic structure. It can be defined by a pair (ω, π), where ω is a 1-form and π is a 2-form of rank $2n$ such that $\omega \wedge \pi^n \neq 0$. It is called a cosymplectic structure if $d\omega = 0$ and $d\pi = 0$.

We quote only one main theorem from the second part: Let ω be a Pfaffian structure on V_{2n+1}. Then there is an isomorphism between the vector space $\mathscr{L}$ of infinitesimal automorphisms of the contact structure induced by ω and the vector space K of differentiable functions on V_{2n+1}. This isomorphism maps the subspace $\mathscr{L}'$ of infinitesimal automorphisms of the Pfaffian structure itself onto the subspace K' consisting of all first integrals of the characteristic system of $d\omega$ (i.e., functions f such that $\theta(E)f = 0$, where E is the vector field associated to ω [cf. Boothby and Wang, Ann. of Math. (2) **68** (1958), 721–734; MR **22** #3015]). The author studies the Lie algebra structure in K induced from the natural Lie algebra structure in $\mathscr{L}$ by this isomorphism, and, conversely, a commutative and associative algebra structure in $\mathscr{L}$ induced from that in K. Various special cases are then studied.

A list of examples given in the last part contains the cosymplectic structures (1) $R \times V_{2n}$ and $S_1 \times V_{2n}$ with symplectic V_{2n}, (2) $\pi = \sum (dp^i \wedge dq^i) - dH \wedge dt$, where H is the Hamilton function and $\omega = dt$. When V_{2n+2} has an almost hermitian structure, it is shown that a regular submanifold V_{2n+1} defined by the equation $f = \text{constant}$ can be provided with an almost cosymplectic structure. As particular cases, there are two such structures on S^5, one of which is Pfaffian.

K. Nomizu (Providence, R.I.)

9920:

Belov, K. M. Some remarks on the rigidity of surfaces of positive curvature with an edge. Dokl. Akad. Nauk SSSR **131** (1960), 475–477 (Russian); translated as Soviet Math. Dokl. **1**, 257–258.

This paper is devoted to a study of continuous bendings of surfaces of positive curvature with an edge in an ordinary Euclidean space. The main result can be stated as follows. Let F_1, F_2 be two isometric surfaces of positive Gaussian curvature with a pair of corresponding edges in an ordinary Euclidean space, b_i, $b_{i\perp}$ $(i = 1, 2)$ the normal curvatures of the surface F_i associated with some direction α_i and the direction perpendicular to α_i respectively; and suppose that $E_i' = \frac{1}{2}(b_i - b_{i\perp})$, $K_i' = b_i b_{i\perp}$. Then the surfaces F_1, F_2 are congruent or symmetric, if one of the following conditions is satisfied. (a) On F_1, F_2 sets of corresponding points, at which $E_1' = E_2'$, $K_1' = K_2'$ and in two fixed corresponding directions, have their limit points on F_1, F_2, respectively. (b) $E_1' = E_2'$, $K_1' = K_2'$ at sets of positive measure of corresponding points on the corresponding edges of F_1, F_2 and in two fixed corresponding directions.

C.-C. Hsiung (Bethlehem, Pa.)

9921:

Godeaux, Lucien. Sur les lignes flecnodales d'une surface réglée. Univ. e Politec. Torino. Rend. Sem. Mat. **18** (1958/59), 159–166.

Utilisant la représentation d'une surface réglée au moyen de la suite de Laplace de la quadrique de Klein

de S_3 dont deux éléments consécutifs sont les images des tangentes asymptotiques de la surface envisagée, l'auteur reprend une étude de G. Tzitzeica consacrée aux surfaces réglées à lignes flecnodales confondues. La suite de Laplace ne contient ici que quatre termes, en présentant d'un côté le cas de Laplace et de l'autre le cas de Goursat. Cette proposition est retrouvée très simplement par l'auteur, qui, entre autres propriétés s'y rattachant, montre que les génératrices des quadriques osculatrices à une surface réglée flecnodale d'une surface réglée, appartenant à la même famille que la tangente flecnodale, engendrent une congruence W, et que, dans le cas où les tangentes flecnodales relatives à chaque génératrice sont confondues, la tangente flecnodale unique g a un contact du quatrième ordre avec la surface réglée envisagée, le complexe linéaire osculateur à cette dernière surface étant spécial (propriété admettant sa réciproque). L'auteur montre en outre que, si pour une surface réglée (x), les tangentes flecnodales sont confondues pour chaque génératrice suivant une droite g, les lignes flecnodales de (g) le sont aussi, et les quadriques osculatrices aux deux surfaces (x) et (g) suivant deux génératrices homologues quelconques sont aussi confondues.

P. Vincensini (Marseille)

9922:

Mambriani, Antonio. Sulla definizione di piano tangente ad una superficie. Ann. Mat. Pura Appl. (4) **49** (1960), 147–156.

L'auteur se borne à traiter le cas fondamental d'une surface S donnée analytiquement en forme explicite $z=f(x, y)$.

Il étend à l'espace à trois dimensions la définition de la tangente à une courbe plane donnée par Tonelli dans ses *Analisi matematica* [Litografia Sacchi, Pisa, 1946]. Pour cela, il introduit d'abord un "voisinage symétrique" d'un plan α_0 appartenant à une étoile de plans $A\{\alpha\}$ de centre A, puis la limite d'un plan variable dans une étoile donnée.

Définition: Un plan τ passant par le point P de la surface S est dit plan tangent à S en P si, un voisinage θ de τ étant pris arbitrairement dans l'étoile de plans $P\{\alpha\}$, il existe un voisinage sphérique Ω de P tel que θ contienne tous les points de S appartenant à Ω.

On introduit ensuite le concept de "tendance freinée angulairement" et on donne les premières conséquences de la définition précédente et les conséquences analytiques; en particulier, on montre que pour qu'en un point P de la surface S, il existe un plan tangent non parallèle à l'axe z, il faut et il suffit que la fonction $f(x, y)$ soit en ce point P, dérivable au sens fort, suivant une définition qui est précisée pour cette expression. L'auteur utilise essentiellement des considérations métriques.

M. Decuyper (Lille)

9923:

Marcus, Froim. On the transformation T of congruences and the definition of a stratifiability of a pair of congruences. Bul. Inst. Politehn. Iaşi (N.S.) **5 (9)** (1959), no. 1–2, 5–12. (Russian and Romanian summaries)

Dans un article des Ann. of Math. (2) **43** (1942), 623–633 [MR **4**, 53], V. G. Grove a étudié les couples de congruences $(\Gamma, \bar{\Gamma})$ de l'espace projectif à trois dimensions en transformation T l'une de l'autre, la transformation T étant définie par les propriétés suivantes: (a) deux rayons correspondants ne sont pas coplanaires, (b) les développables des deux congruences se correspondent, (c) il existe pour chaque congruence trois surfaces transversales au moins dont les plans tangents en trois points situés sur un même rayon passent par le rayon correspondant de l'autre congruence. Il est ainsi amené á considérer deux types (asymptotique et conjugué) de transformations T, dont il étudie les propriétés, et notamment les circonstances relatives au case où les congruences envisagées sont W, et, dans le cas de conjugaison, le comportement du quadrilatère formé par les deux foyers de l'un quelconque des rayons de l'une Γ des deux congruences et deux points situés sur le rayon correspondant de l'autre.

Etudiant de plus près la configuration $(\Gamma, \bar{\Gamma})$, l'auteur, dans une note des mêmes Ann. **54** (1951), 552–553 [MR **13**, 689] a énoncé, sans démonstration, le fait que les transformations T ne donnent lieu qu'à la considération d'un seul type, soit le type conjugué, et qu'alors (complétant ainsi un résultat de V. G. Grove) sont simultanément W avec Γ et $\bar{\Gamma}$ les congruences engendrées par les différents côtés du quadrilatère auquel il a été fait allusion plus haut. L'auteur, s'appuyant sur certains résultats de G. Fubini et S. Finikoff, donne ici la démonstration complète de cette affirmation.

P. Vincensini (Marseille)

9924:

Berezina, L. Some properties of a Γ congruence. Latvijas PSR Zinātņu Akad. Vēstis **1960**, no. 6 (155), 57–60. (Russian. Latvian summary)

Une congruence Γ est une congruence de droites dont les développables correspondent aux lignes de courbure des surfaces focales [S. P. Finikow, *Theorie der Kongruenzen*, Akademie-Verlag, Berlin, 1959; MR **21** #5212; p. 95]. L'auteur établit, par le calcul, deux théorèmes; le premier s'énonce: "Les lignes de courbure des surfaces focales d'une congruence orthogonales aux rayons de la congruence ont, aux points correspondants, des rayons de courbure géodésique égaux à la distance entre les foyers"; le second est relatif aux rayons de courbure normale des lignes focales. On peut préciser l'énoncé du premier théorème: le centre de courbure géodésique à la courbe considérée en un foyer est l'autre foyer, et l'on peut donner une démonstration géométrique directe de cette proposition.

M. Decuyper (Lille

9925:

Redozubova, O. S. Special kinds of T-pairs of congruence. Dokl. Akad. Nauk SSSR **137** (1961), 284–286 (Russian); translated as Soviet Math. Dokl. **2**, 276–278.

Investigated are those T-pairs of rectilinear congruences in ordinary three-space [S. P. Finikov, *Teoriya par kongruèncii*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956; MR **19**, 676] for which on corresponding rays the focal distances and the angles between the focal planes are equal. There are two types, the first type depending on four functions of one argument, the second type on one function of two arguments. Among the properties of the first type is $\rho_1=\rho_2$, $\rho_1'=\rho_2'$, among those of the second type $\rho_1=-\rho_2'$, $\rho_2=-\rho_1'$; here ρ_k, ρ_k', $k=1, 2$, are the abscissae of the foci F_k, F_k' of the congruences, measured from the points of intersection with the rays of the congruence of mutual perpendiculars. A number of theorems

are derived for these two types; for instance, a necessary and sufficient condition that in the case of normal congruences the angles and the distances between corresponding rays are constant is that on every ray the product $\rho_1\rho_2$ be constant. *D. J. Struik* (Cambridge, Mass.)

9926:

Hsiao, E. K. On the theory of Löbell on transformation of surfaces. Amer. Math. Monthly **67** (1960), 353–356.

Geometric interpretations are given of some mixed expressions of the coefficients of the first fundamental forms of two parallel surfaces in an ordinary Euclidean space, one of the surfaces being of constant Gaussian curvature. *C. C. Hsiung* (Bethlehem, Pa.)

9927:

Calapso, Maria Teresa. Intorno ad una trasformazione delle equazioni paraboliche. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **27** (1959), 202–204.

In a preceding note [same Atti (8) **26** (1959), 757–762; MR **22** #4995] the authoress has been concerned with surfaces of the form (1) $x=x(u, v)$ (indices suppressed) upon which the lines $v=$const are asymptotic, so that the n functions $x(u, v)$ satisfy equations of the form (2) $x_{uu}=ax_u+bx_v$. A parabolic congruence of which the surface (1) is the unique focal surface is described by the line (3) $\bar{x}=x+tx_u$ and the conditions are given in order that the parabolic congruence should be conjugate to the family of asymptotic lines and vice versa. The passage from any asymptotic family to one which is conjugate to the parabolic congruence is called a C-transformation, and this gives rise to a corresponding transformation, also called a C-transformation, in the parabolic equation (2) to one of the same type.

In this note the study of the C-transformations is carried a stage further with emphasis laid on some related geometrical facts. A spherical representation of the congruence (3) is introduced, and in terms of this spherical representation the main result is: Any asymptotic family conjugate to a parabolic congruence is always parallel to an asymptotic family of the spherical representation. *E. T. Davis* (Southampton)

9928:

Ivanović, Dragiša M. ★Vektorska analiza [Vector analysis]. University of Belgrade, Belgrade, 1960. vii+354 pp. $6.00.

A textbook for engineers and physicists as well as for students of physics and technics. It consists of eight parts, from vector algebra to linear operators, and is well fitted for initiation to vectorial methods. It contains many solved problems and exercises.

The book is on the level of similar literature anywhere and standard notations are used. Technically it is well put together. *T. P. Anđelić* (Belgrade)

9929:

Johnson, H. H. Terminating prolongation procedures. Pacific J. Math. **10** (1960), 577–583.

The paper concerns applications of prolongation theorems of exterior differential systems. Let S be a system on $(x^1, \cdots, x^p, y^1, \cdots, y^n)$, where $x^1, \cdots, x^p$ are independent variables, generated by 1-forms $\theta^1, \cdots, \theta^m$ (which are linearly independent mod $\omega^i=dx^i$) and by their exterior derivatives. Take 1-forms $\pi^1, \cdots, \pi^{n-m}$ to form a base with the θ^α, ω^i. We assume that $d\theta^\alpha \equiv A_{i\lambda}{}^\alpha \omega^i \wedge \pi^\lambda + \frac{1}{2}B_{ij}{}^\alpha \omega^i \wedge \omega^j \pmod{\theta^\beta}$, with $B_{ij}{}^\alpha + B_{ji}{}^\alpha = 0$. Denote by $P^l(S)$ the lth prolongation of S. Theorem 1 says: If (1) $A_{i\lambda}{}^\alpha$ is a constant, (2) $dB_{ij}{}^\alpha \equiv 0 \pmod{\omega^k\theta^\beta}$, and (3) integral points of $P^l(S)$ are ordinary integral points, then, roughly speaking, $P^l(S)$ is an involution for large l or S has no solution. {However, these conditions are not sufficient and we have to replace (2) by a stronger condition in order for Theorem 1 to be true.} Theorem 2 is a special case of Theorem 1 when $n=m$, and in this special case Theorem 1 is correct. Next the author explains the process of what might be called the adding of functions in the prolongated systems and shows that the reviewer's generalized prolongation theorem can be applied to obtain an involutive system by this process provided the system satisfies certain regularity conditions.

M. Kuranishi (Princeton, N.J.)

9930:

Romain, Jacques. Invariants différentiels d'un espace non riemannien. C. R. Acad. Sci. Paris **244** (1957), 2777–2779.

Author's summary: "Nous calculons le nombre des invariants différentiels indépendants des divers ordres d'un espace à n dimensions non riemannien (espace défini par un tenseur fondamental du second ordre asymétrique et une connexion affine asymétrique indépendante de ce tenseur). Nous obtenons une première indication de la structure de ces invariants."

9931:

Romain, Jacques. Invariants différentiels d'un tenseur asymétrique du second ordre à quatre dimensions. C. R. Acad. Sci. Paris **251** (1960), 1336–1338.

In #9930 the author merely counted the number of differential invariants of mth order of an asymmetric tensor g_{ik} $(i, k=1, 2, \cdots, n)$. For $n=4$ there are 24 differential invariants of the first order. The author quotes a system of 15+9 invariants, which are said to be independent. No explicit proof of the completeness, or of a relation to the projective invariants of a quadric, a linear complex and two quadratic complexes, is given.

E. M. Bruins (Amsterdam)

9932:

Romain, Jacques. Invariants différentiels d'un champ maxwellien. C. R. Acad. Sci. Paris **251** (1960), 1975–1977.

Splitting up the asymmetric tensor g_{ik} $(i, k=1, 2, 3, 4)$ into a symmetric part γ_{ik} and an alternating part φ_{ik}, the 24 differential invariants of the first order [given in #9931] are considered in the special case that $\gamma_{11}=\gamma_{22}=\gamma_{33}=-\gamma_{44}=-1$, $\gamma_{ik}=0$ for $i\neq k$, whereas φ_{ik} is a Maxwell tensor corresponding to electromagnetic fields in vacuo. It turns out that all but four invariants vanish and that these four are reducible to one single invariant I. The author demonstrates some relations between I and the radii of curvature and torsion of curves on the wave-front surface. *E. M. Bruins* (Amsterdam)

9933:

Georgiu, O. È. [Gheorghiu, O. E.] On the theory of two-component special geometric objects. Rev. Math. Pures Appl. **4** (1959), 77–93. (Russian)

Es werden als Fortsetzung der Arbeit des Verf. in Com. Acad. R.P. Romîne **2** (1952), 1–4 [MR **18**, 596] geometrische Objekte mit zwei Komponenten in einem beliebigen Punkte P_0 des n-dimensionalen Raumes mit Transformationsformeln der Gestalt

$$W_1' = W_1 + A, \tag{1}$$

$$W_2' = W_2 + (B/A)(e^A - 1)e^{W_1} \tag{2}$$

gesucht, wo A, B (und im folgenden C, f, k) zu bestimmende (stetige oder differenzierbare) Funktionen sind, die von der Transformation des Koordinatensystems x^i in das mit den Koordinaten $x^{i'}$ ($i, i' = 1, 2, \cdots, n$) abhängen, deren Gestalt aber gegenüber Koordinatentransformationen invariant ist, und W_1, W_2 bzw. W_1', W_2' die Komponenten des Objektes in diesen beiden Koordinatensystemen bedeuten. Das Problem würde mehrfache Reduktion ermöglichen. Vor allem läßt sich (2) natürlich einfacher als

$$W_2' = W_2 + Ce^{W_1} \tag{2'}$$

schreiben. Die Abhängigkeit der A, B, C von der Koordinatentransformation wird durch Abhängigkeit von den durch die Koordinaten und Ableitungswerte im Punkte P_0 gegebenen Parametern

$$x = \{x^i_0\}, \qquad x' = \{x^{i'}_0\},$$

$$D = \left\{\left(\frac{\partial x^{i'}}{\partial x^i}\right)_0, \cdots, \left(\frac{\partial^r x^{i'}}{\partial x^{i_1} \cdots \partial x^{i_r}}\right)_0\right\}$$

vorausgesetzt (wo A eventuell von einigen der letzten Parametergruppen unabhängig ist). Nun kann man aber von der Abhängigkeit von den Parametergruppen x, x' absehen, da für A und C offenbar

$$A(x, c, E) + A(c, x, E) = 0,$$

$$C(x, c, E) + C(c, x, E)e^{A(x,c,E)} = 0$$

(c konstant; $E = \{\delta_i^j, 0, \cdots, 0\}$; $\delta_i^j = 1$ für $i = j$, $= 0$ für $i \neq j$),

$$A(x, x', D) = A(x, c, E) + A(c, c, D) + A(c, x', E),$$

$$C(x, x', D) = C(x, c, E) + C(c, c, D)e^{A(x,c,E)} + C(c, x', E)e^{A(x,c,E)+A(c,c,D)},$$

also — mit $A(c, c, D) = A(D)$, $C(c, c, D) = C(D)$, $A(x, c, E) = a(x)$, $C(x, c, E)e^{-a(x)} = b(x)$ —

$$\begin{aligned} A(x, x', D) &= A(D) + a(x) - a(x'), \\ C(x, x', D) &= (C(D) + b(x) - b(x')e^{A(D)})e^{a(x)} \end{aligned} \tag{3}$$

gelten. (So kann man die vom Verf. an S. 93 zitierten Ergebnisse über rein differentielle Objekte doch anwenden.) Da Verf., wie in seinen anderen Arbeiten, auch hier diese Reduktionsmöglichkeit nicht beachtet, werden seine Rechnungen etwas verwickelt. Eine weitere vom Verf. unbeachtete Reduktionsmöglichkeit: Man bezeichne

$$O_1 = e^{W_1}, \quad O_2 = W_2 e^{-W_1}, \quad f = e^A, \quad k = e^{-A}C \tag{4}$$

und erhält den Spezialfall

$$O_1' = fO_1, \tag{1''}$$

$$O_2' = f^{-1}O_2 + k \tag{2''}$$

der vom Verf. und B. Crstici [Acad. R. P. Romîne. Stud. Cerc. Mat. **9** (1958), 311–331; MR **21** #7513] untersuchten Transformationsformeln. Nach der obigen Elimination von x, x' kann man die Bestimmung der Objekte mit den Transformationsformeln (1''), (2'') bei S. Goląb und M. Kucharzewski [Tensor (N.S.) **8** (1958), 1–7; MR **20** #4283], M. Kuczma [Publ. Math. Debrecen **6** (1959), 72–78; MR **21** #5976] und M. Kucharzewski und M. Kuczma [Tensor (N.S.) **10** (1960), 245–254; **11** (1961), 35–42] ohne irgendwelche Regularitätsannahmen über f und k finden. Verf. setzt (stillschweigend) z.T. Stetigkeit, z.T. Differenzierbarkeit voraus. Wenn nur z.B. Meßbarkeit vorausgesetzt wird (auch das muß nicht immer vorausgesetzt werden), so kommen unter der obigen Reduktion im wesentlichen nur

$$O_1' = O_1,$$

$$O_2' = O_2;$$

$$O_1' = O_1,$$

$$O_2' = O_2 + a \log \left| \det \left(\frac{\partial x^{i'}}{\partial x^i}\right)_0 \right|;$$

$$O_1' = \left| \det \left(\frac{\partial x^{i'}}{\partial x^i}\right)_0 \right|^a O_1,$$

$$O_2' = \left| \det \left(\frac{\partial x^{i'}}{\partial x^i}\right)_0 \right|^{-a} O_2 + b\left(\left| \det \left(\frac{\partial x^{i'}}{\partial x^i}\right)_0 \right|^{-a} - 1\right);$$

und im eindimensionalen Raume noch

$$O_1' = \left(\frac{dx'}{dx}\right)_0^2 O_1,$$

$$O_2' = \left(\frac{dx'}{dx}\right)_0^{-2} O_1 + a\left(\left(\frac{d^3x'}{dx^3}\right)_0 \left(\frac{dx'}{dx}\right)_0^{-3} - \frac{3}{2}\left(\frac{d^2x'}{dx^2}\right)_0^2 \left(\frac{dx'}{dx}\right)_0^{-4}\right) + b\left(\left(\frac{dx'}{dx}\right)_0^{-2} - 1\right)$$

als Transformationsformeln (1''), (2'') in Betracht, und weitere gibt es nicht. Mittels (3) und (4) erhält man die Objekte mit Transformationsformeln der Gestalt (1), (2') (oder (2)) und das enthält die Ergebinisse der vorliegenden Arbeit.

Dies bestimmt auch gewisse Korrigenda in diesem Artikel. So sollte—falls nicht nur positive φ' zugelassen werden—in der zweiten Formel von (23) $|\varphi'|^m$ statt φ'^m stehen (auch in (22) usw.) und (33) ist nur für $\varphi' > 0$ ein Objekt ($\varphi' = (dx'/dx)_0$). Auf den Seiten 88–92 ist es nicht klar ob $|p_i{}^a|$ die $\det p_i{}^a$ oder $|\det p_i{}^a|$ bedeutet (das letztere ist richtig), usw. Ref. konnte sich von der Richtigkeit des Nichtexistenzbeweises auf S. 91–92 nicht überzeugen. Es preist jedenfalls die Rechengeschicklichkeit des Verfs., daß er seine unreduzierten, verwickelten Rechnungen mit wesentlich richtigen Ergebnissen durchführt. *J. Aczél* (Debrecen)

9934:

Marcus, F. Sur des classes de surfaces qui admettent ∞^3 congruences associées à une congruence de droites associable relativement à ces surfaces. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. Mat. **10** (1959), 307–326. (Romanian. Russian and French summaries)

Given a surface S in projective 3-space. Through each point of S construct a straight line which does not lie in the tangent plane of S at that point. The resulting congruence Γ together with S defines two quadratic differential forms. If another congruence of this type exists which combined with S defines the same forms, Γ is called associable [Terracini, Ann. Scuola Norm. Sup. Pisa (2) **2** (1933), 401–428].

The author studies surfaces which have associable congruences conjugate to them and consisting of canonical straight lines. "Nous étudions encore la figure formée d'une même surface et de deux congruences telles que les courbes d'intersection des développables d'une congruence avec la surface soient conjuguées avec les courbes d'intersection des développables de l'autre congruence avec la surface." *P. Scherk* (Toronto)

9935:

Rešetnyak, Yu. G. On conformal mappings of a space. Dokl. Akad. Nauk SSSR **130** (1960), 1196–1198 (Russian); translated as Soviet Math. Dokl. **1**, 153–155.

The author generalizes Liouville's theorem on conformal mapping of a space in such a way as to dispense with assumptions of differentiability. He considers a topological mapping of an (open) domain M in n-dimensional Euclidean space, with $n \geq 3$, and supposes it to be conformal in the sense that, roughly speaking, for each point x in M a family of approximately spherical (nested) small neighbourhoods of x is transformed into another such family of neighbourhoods of the image of x. He finds that these assumptions suffice to make the mapping analytic, so that, by the classical result mentioned above, the mapping must be a combination of a finite number of inversions. He indicates briefly the scheme of his proof, including enunciations of lemmas.

H. P. Mulholland (Exeter)

9936:

Sigalov, A. G. Conditions for the compactness of a family of surfaces of finite topological type. Dokl. Akad. Nauk SSSR **130** (1960), 1203–1205 (Russian); translated as Soviet Math. Dokl. **1**, 158–160.

In this note, the author generalizes certain parts of the apparatus of his recent memoir [Uspehi Mat. Nauk **12** (1957), no. 1 (73), 53–98; Amer. Math. Soc. Transl. (2) **14** (1960), 59–106; MR **19**, 560; **22** #3999]. The underlying space is now a manifold with a Riemann metric and the surfaces concerned are specified by parametric representations $x(u)$, absolutely continuous in the Tonelli sense and with finite Dirichlet integral, defined on the unit square with identifications. Equivalence of representations can be either in the Fréchet sense, or in the elementary sense. Four compactness theorems are listed for a family of such surfaces, which have uniformly bounded areas and are situated in a bounded part of the underlying space. These theorems link compactness to a property, introduced in effect in the memoir, which he terms "uniform non-degenerateness". Indications as to the proofs ignore the oversight in the memoir, which the author attempted to rectify in a subsequent note [Uspehi Mat. Nauk **15** (1960), no. 1 (91), 261; MR **22** #4000].

L. C. Young (Madison, Wis.)

9937:

Kyner, Walter T. Invariant manifolds. Bol. Soc. Mat. Mexicana (2) **5** (1960), 75–76.

Some results are announced on the existence of invariant manifolds of homeomorphisms of a Riemannian manifold onto itself. The proofs and applications to ordinary differential equations are to appear elsewhere.

J. K. Hale (Baltimore, Md.)

9938:

Nomizu, Katsumi. On local and global existence of Killing vector fields. Ann. of Math. (2) **72** (1960), 105–120.

L'auteur utilise le système différentiel vérifié par l'opérateur de dérivation associé à un champ de vecteurs de Killing sur une variété riemannienne [voir B. Kostant, Trans. Amer. Math. Soc. **80** (1955), 528–542; MR **18**, 930] pour étudier le prolongement des germes de tels champs. Il établit en particulier que, sur une variété riemannienne analytique simplement connexe M, l'algèbre de Lie des champs de vecteurs de Killing est isomorphe à une sous-algèbre de l'algèbre de Lie des dérivations de l'espace tensoriel attaché à un point quelconque x de M. Application: condition d'existence d'un groupe transitif d'isométries sur une variété riemannienne complète simplement connexe. *J. Lelong* (Paris)

9939:

Tamássy, L. Über die geometrische Anwendung eines Differentialgleichungssystems. Acta Math. Acad. Sci. Hungar. **11** (1960), 187–195. (Russian summary, unbound insert)

The classical procedure (J. M. Thomas, O. Veblen) for the equivalence of two affinely connected spaces (existence of a coordinate transformation changing the parameters of one connection to those of the other) leads to a system of partial differential equations. The author observes (and gives examples thereof) that the usual criterion is not always sufficient for the equivalence (depending on the impossibility of inverting the existing solution). A necessary and sufficient condition can be given by adding to the differential system a system of scalar equations.

E. Bompiani (Rome)

9940:

Ulanovskiĭ, M. A. Stationary groups of motions of spaces with an affine connection. Dokl. Akad. Nauk SSSR **131** (1960), 34–36 (Russian); translated as Soviet Math. Dokl. **1**, 196–199.

In a manifold M with affine connection let S be the group of motions which keeps a point O of M fixed. By means of normal coordinates one can establish a one-to-one correspondence between the points of a neighborhood of O and points in a neighborhood of the tangent space E of M at O. Linear subspaces of E thus correspond to local submanifolds of M and S acts also on E. Under various hypotheses on the linear subspaces which are related to S, the corresponding local submanifolds are proved to be totally geodesic. *G. B. Allendoerfer* (Seattle, Wash.)

9941:

Sabinin, L. V. Explicit expression of the forms of connection of a quasisymmetric space in terms of the values of the curvature and torsion tensors at some point.

Dokl. Akad. Nauk SSSR **132** (1960), 1273–1276 (Russian); translated as Soviet Math. Dokl. **1**, 778–781.

The background of this problem can be most easily found in a paper by K. Nomizu [Amer. J. Math. **76** (1954), 33–65; MR **15**, 468], although the author does not refer to it. A quasisymmetric space is a manifold with an affine connection whose torsion and curvature tensors have vanishing covariant derivative. It is known that such a space is locally equivalent to a homogeneous space G/K with an invariant affine connection, G a connected Lie group, K a Lie subgroup, that the geodesics are orbits of one-parameter subgroups of G, that the curvature and torsion tensors can be described in terms of the Lie algebras of G and K, and that the connection forms can be described in terms of the invariant Cartan-Maurer forms on G. A classical result of E. Cartan [see C. Chevalley, *Theory of Lie groups. I*, Princeton Univ. Press, Princeton, N.J., 1946; MR **18**, 583; p. 157] shows how the Cartan-Maurer forms can be computed in a canonical coordinate system for G. The normal geodesic coordinates of the affine connection on G/K can be considered as the coordinates of a linear subspace of the canonical coordinates for G.

Putting these known facts together gives in principle the components of the connection form of the affine connection on G/K in a normal geodesic coordinate system. The author carries out these computations explicitly in terms of a basis for the Lie algebra of G.

R. Hermann (Belmont, Mass.)

9942:

Vrănceanu, G. G. Transformations ponctuelles à connexion projective constante. Rev. Math. Pures Appl. **4** (1959), 157–160.

Given a point-to-point correspondence between two projective planes, the author has already shown the existence of a tensor associated with this correspondence [Boll. Un. Mat. Ital. (3) **12** (1957), 489–506; MR **19**, 1075]. Here he gives the canonical forms of a correspondence whose associated tensor has constant components.

E. Bompiani (Rome)

9943:

Woinaroski, R. Rigid motions with fixed point in euclidean space of five dimensions. I, II. Lucrările Inst. Petrol Gaze Geol. București **5** (1959), 383–396, 397–406. (Romanian. Russian and English summaries)

This is the sequel to an earlier paper by the same author concerned with rigid motions with fixed point in the euclidean 5-space. In such a motion one of the planes of instantaneous rotation generates a Riemannian space of three dimensions in the fixed space as well as in the mobile space. By calculating the components of the curvature the author is able to show that these two spaces are of the same nature, i.e., either both curved (the general case) or flat.

R. Blum (Kingston, Ont.)

9944:

Busemann, Herbert. On spaces with one-parameter groups of motions. Tensor (N.S.) **10** (1960), 21–25.

Continuing his study of one-parameter groups of motions in G-spaces [cf. *The geometry of geodesics*, Academic Press, New York, 1955; J. Math. Pures Appl. (9) **37** (1958), 365–373; MR **17**, 779; **21** #913], the author derives by his synthetic methods various relations on the structure of spaces admitting such groups. Theorem: Let the G-space R have a finitely generated fundamental group and a one-parameter group of motions, P_t. If P_t possesses a closed orbit, a multiple of this orbit represents an element in the center of the fundamental group. Now let R be compact and without conjugate points (i.e., the universal covering space is straight). Then the orbits of any one-parameter group of motions of R are simple geodesics and any closed orbit contains in a suitable multiple of its free homotopy class a family of simple closed geodesics, all of the same length, covering R simply. If R is Riemannian, its universal covering space is the cartesian product (topologically and metrically) of a straight subspace and a straight line. Corollary: A compact Riemann space without conjugate points has only a finite number of motions, unless every point lies on a line element L such that the sectional curvature vanishes for every plane element containing L. This is related to a theorem of Bochner's [Bull. Amer. Math. Soc. **52** (1946), 776–797; MR **8**, 230; cf. also R. Hermann, #9945]. In the course of proving the main theorem, some previous results on G-spaces are sharpened; for example: the fundamental group of a compact G-space with strictly convex capsules and domain invariance does not have a center. This includes Riemann spaces with negative curvature [cf. Švarc, Dokl. Akad. Nauk SSSR **105** (1955), 32–34; MR **17**, 781].

L. W. Green (Minneapolis, Minn.)

9945:

Hermann, Robert. On geodesics that are also orbits. Bull. Amer. Math. Soc. **66** (1960), 91–93.

The author observes that if X is a Killing vector field on a Riemannian manifold M, and g denotes the function defined on the unit sphere bundle of M which assigns to $v \in M_p$ the inner product $\langle X_p, v_p \rangle$, then v_p is a critical point of g if and only if the geodesic through v_p is an orbit of X. He gives a geometric interpretation of the index of such a critical point and makes various applications of his theorem.

R. Bott (Cambridge, Mass.)

9946:

Saban, Giacomo. Sopra una caratterizzazione della sfera. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **28** (1960), 345–349.

Let s, ρ_n, ρ_g, τ_g denote the arc length, normal curvature, geodesic curvature, and geodesic torsion respectively of a curve $\mathscr{L}$ on a surface Σ. If N denotes the normal vector of Σ at the point x, the Darboux function is defined on $\mathscr{L}$ through

$$\mathscr{D} = \mathscr{D}(s) = \frac{d^2\mathbf{x}}{ds^2} \cdot \mathbf{N} = \frac{d\rho_n}{ds} + \rho_g \tau_g.$$

Two proofs of the following result are given: Σ is (part of) a sphere or plane if and only if $\oint_{\mathscr{L}} \mathscr{D} ds = 0$ for every $\mathscr{L}$ in Σ.

P. Scherk (Toronto)

GENERAL TOPOLOGY, POINT SET THEORY

See also 9545, 9847, 9970.

9947:

Aleksandrov, P. S. Some results in the theory of topological spaces obtained within the last twenty-five years. Uspehi Mat. Nauk **15** (1960), no. 2 (92), 25–95 (Russian); translated as Russian Math. Surveys **15** (1960), no. 2, 23–83.

A clear, although sometimes incomplete (as pointed out by the author) survey is given of the development and main results in some selected topics in general topology. Contents: compactness and related properties; properties defined in terms of locally finite, star-finite, etc., coverings; metrizability; products of spaces; interrelations of uniform, proximity and topological spaces; extension of spaces (compact and H-closed); perfect mappings and related questions; multivalued mappings; extension of mappings; some results in general dimension theory; inverse systems of coverings or of polyhedra and related questions; dimension theory for infinite-dimensional spaces; Cantor manifolds. The bibliography contains 144 items. *M. Katětov* (Prague)

9948:

Brown, Morton. On the inverse limit of Euclidean N-spheres. Trans. Amer. Math. Soc. **96** (1960), 129–134.

Let $S=\lim(S_i^N, f_i)$ denote the inverse limit of N-spheres S_i^N with bonding maps $f_i: S_i^N \to S_{i-1}^N$ (N fixed, $i=1, 2, \cdots$). When $N=1$, Bing has constructed [Pacific J. Math. **1** (1951), 43–51; MR **13**, 265] an S which is hereditarily indecomposable (h.i.) and such that each f_i is of degree 1. The author shows here that no such theorem is possible if $N>1$. He proceeds by showing first that if any inverse limit $\lim(X_i, f_i)$ is to be h.i., then given $\varepsilon>0$ there exists n such that $f_1 \circ f_2 \circ \cdots \circ f_n$ is what Bing calls "ε-crooked". But, when $N>1$, he shows that there is an $\varepsilon>0$ such that no essential map of S^N onto itself is ε-crooked; the proof uses the fact that S^N is unicoherent if $N>1$. Hence S, an inverse limit of N-spheres with the f_i essential, cannot be h.i., if $N>1$. *H. B. Griffiths* (Birmingham)

9949:

Henderson, George W. Proof that every compact decomposable continuum which is topologically equivalent to each of its nondegenerate subcontinua is an arc. Ann. of Math. (2) **72** (1960), 421–428.

In the early 1920's topologists were much interested in characterizations of the compact simple continuous curves (simple closed curves and arcs). While several such characterizations were obtained, two very intriguing possibilities were formulated: (1) if a compact nondegenerate plane continuum is homogeneous, is it a simple closed curve and (2) if a compact nondegenerate plane continuum is homeomorphic to each of its nondegenerate subcontinua, is it an arc? Except to observe that the answer is "yes" to both questions in the presence of local connectivity, no substantial progress was made for more than twenty-five years. Interest was revived by E. E. Moise's invention of the pseudo-arc (a very crooked sort of indecomposable continuum). Moise showed that it gave a negative answer to (2) and Bing (followed closely by Moise) showed that it gave a negative answer to (1). The characterization questions now became classification questions. The reviewer showed that in the presence of decomposability a rather natural monotone decomposition of the homogeneous continuum is a simple closed curve [Proc. Amer. Math. Soc. **6** (1955), 735–740; MR **17**, 180] but the continuum itself need not be [Bing and Jones, Trans. Amer. Math. Soc. **90** (1959), 171–192; MR **20** #7251]. However, this leaves the classification problem arising from (1) unsolved even for decomposable continua. On the other hand, with the result announced in the title of this paper the author has, for decomposable continua, completed the solution of the classification problem arising from (2). The case for indecomposable continua remains open. *F. B. Jones* (Chapel Hill, N.C.)

9950:

Kirtadze, G. A. Different types of completeness of topological spaces. Mat. Sb. (N.S.) **50 (92)** (1960), 67–90. (Russian)

The author investigates spaces complete with respect to adjoining points satisfying some of the separation axioms T_1, T_2, T_3 or the complete regularity axiom T_ρ (T_2-complete T_2-spaces coincide with H-closed spaces, T_ρ-complete T_ρ-spaces with compact Hausdorff ones). Various necessary and sufficient conditions are given for the T_i-completeness ($i=1, 2, 3, \rho$), and several examples are constructed, including that of a non-H-closed Hausdorff space such that, for any infinite subset M, there is a point x for any neighborhood U of which the potency of $\overline{U} \cap M$ is equal to that of M. *M. Katětov* (Prague)

9951:

Frolík, Z. Locally complete topological spaces. Dokl. Akad. Nauk SSSR **137** (1961), 790–792 (Russian); translated as Soviet Math. Dokl. **2**, 355–357.

This concerns local absolute G_δ's. If paracompact, they are absolute G_δ's. [Cf. E. Michael, Duke Math. J. **21** (1954), 163–171; MR **15**, 977.] In any case there is a dense open subset which is an absolute G_δ. In these spaces the character and pseudocharacter of a point coincide. Also, a product of two pseudocompact spaces, one of which is a local absolute G_δ, is pseudocompact. The property is preserved backward and forward by perfect (proper) mappings, and forward by open mappings.
J. R. Isbell (Seattle, Wash.)

9952:

Kowalsky, Hans-Joachim. Verbandstheoretische Kennzeichnung topologischer Räume. Math. Nachr. **21** (1960), 297–318.

Die Topologie τ eines aus der Trägermenge R und dieser Topologie bestehenden topologischen Raumes (R, τ) ist bekanntlich durch das System $\mathfrak{y}(R, \tau)$ aller offenen Teilmengen von (R, τ) eindeutig bestimmt. $\mathfrak{y}(R, \tau)$ ist hinsichtlich der mengentheoretischen Inklusion ein vollständiger Verband, der als der Strukturverband von (R, τ) bezeichnet wird. Homöomorphe Räume besitzen isomorphe Strukturverbände, während aus der Isomorphie der letzteren auf die Homöomorphie der Räume im Allgemeinen nicht geschlossen werden darf. Die in der vorliegenden Arbeit behandelten Probleme betreffen die verbandstheoretische Charakterisierung von Strukturverbänden, die Bestimmung aller topologischen Räume

mit zu einem gegebenen Verband isomorphen Strukturverbänden; ferner handelt es sich um Untersuchung von Fällen in denen isomorphe Strukturverbände auf homöomorphe Räume führen und um Betrachtungen über die sogenannten verwandtschaftstreuen Eigenschaften, d.h. Eigenschaften, die gegenüber der durch die Isomorphie von Strukturverbänden definierten Äquivalenz aller topologischen Räume invariant sind. Die Arbeit enthält zahlreiche neue und tiefliegende Resultate, die zum Teil mit dem Begriff eines speziellen Filtertypus, dem der sogenannten Trichter, eng zusammenhängen.

O. Borůvka (Brno)

9953:

Pasynkov, B. A class of transitive single-valued spectra for bicompacta. Dokl. Akad. Nauk SSSR **131** (1960), 253–254 (Russian); translated as Soviet Math. Dokl. **1**, 235–236.

In this paper a spectrum $S=\{X_\alpha, \omega_\alpha^\beta\}$ is a system of finite simplicial complexes X_α (not required to contain all faces of their simplexes) and simplicial maps $\omega_\alpha^\beta: X_\beta \to X_\alpha$, defined uniquely whenever $\alpha<\beta$ (α, β range over a directed set A). The maps are subject to a transitivity condition: $\omega_\alpha^\beta \omega_\beta^\gamma = \omega_\alpha^\gamma$, for $\alpha<\beta<\gamma$. Each complex X_α can be considered as ordered by writing $t_\alpha' \leqq_\alpha t_\alpha$ when t_α' is a face of the simplex $t_\alpha \in X_\alpha$. The direct product $\prod X_\alpha$, $\alpha \in A$, is then ordered by the induced product order $\leqq$. A projection set $t=\{t_\alpha\}$ is an element of $\prod X_\alpha$ such that $t_\alpha = \omega_\alpha^\beta t_\beta$, for $\alpha<\beta$. Finally a thread (filament) is a projection set, minimal with respect to the order $\leqq$. With each spectrum S a limit space X is associated. The points of X are threads $t=\{t_\alpha\}$ of S. Neighbourhoods of $t \in X$ are sets $U_{\alpha_0} t$ consisting of all the threads $t'=\{t_\alpha'\}$ for which t_{α_0}' belongs to the open star of t_{α_0} in X_{α_0}.

If S satisfies a certain Hausdorff condition, then X is always a bicompactum [P. S. Aleksandrov, Uspehi Mat. Nauk **2** (1947), no. 1 (17), 5–57; MR **10**, 389]. Among all the Hausdorff spectra S the author singles out a class of "extremal spectra" Σ, characterized by certain maximality conditions. E.g., the spectrum Σ should not be obtainable from some other spectrum by a weakening of the order. The main theorem asserts that there is a one-to-one correspondence between bicompacta X and extremal spectra Σ. No proofs are given.

S. Mardešić (Zagreb)

9954:

Arhangel'skiĭ, A. External bases of sets lying in bicompacta. Dokl. Akad. Nauk SSSR **132** (1960), 495–496 (Russian); translated as Soviet Math. Dokl. **1**, 573–574.

The author sharpens lemmas of Yu. Smirnov [Fund. Math. **43** (1956), 387–393; MR **18**, 813] and concludes in particular that a countable union of closed separable metrizable spaces is metrizable if it is an absolute G_δ. Also, a continuous mapping onto an absolute G_δ cannot increase weight.

J. R. Isbell (Seattle, Wash.)

9955:

Stone, A. H. Sequences of coverings. Pacific J. Math. **10** (1960), 689–691.

The following criterion for metrizability of a T_0-space S is given: There exist open coverings $\mathscr{U}_n$ ($n=1, 2, \cdots$) such that, for any $x \in S$, the collection of all $\mathrm{St}(G, \mathscr{U}_n)$, G open, $x \in G$, is a neighborhood base at x. It is shown that a completely regular S is metrizable if and only if there exist a uniformity for S and a sequence $\{\mathscr{V}_n\}$ of (possibly non-uniform) open coverings such that every finite uniform covering is refined by some $\mathscr{V}_n$. There are other analogous results, especially concerning metrizable spaces for which the set of limit points is compact.

M. Katětov (Prague)

9956:

Aleksandrov, P. S.; Ponomarev, V. I. Compact extensions of topological spaces. Vestnik Moskov. Univ. Ser. Mat. Meh. Astr. Fiz. Him. **1959**, no. 5, 93–108. (Russian)

Consider a binary relation, $A<B$, defined for subsets of a topological space. If certain axioms are satisfied, then this relation is called a "subordination" (for instance, the "elementary subordination" is defined for a completely regular X as follows: $A<B$ if and only if A and $X-B$ are functionally separated). It turns out that this notion is equivalent, in a natural way, to that of a proximity relation. Various results from the theory of proximity spaces are formulated and proved by means of "subordination", as well as several new theorems (concerning, e.g., topological spaces possessing open bases consisting of sets with compact boundaries).

M. Katětov (Prague)

9957:

Taĭmanov, A. D. On closed mappings. II. Mat. Sb. (N.S.) **52 (94)** (1960), 579–588. (Russian)

The author has shown [Mat. Sb. (N.S.) **36 (78)** (1955), 349–352; MR **17**, 287] that closed mappings in separable metric spaces take Borel sets to Borel sets. I. A. Vaĭnšteĭn has shown [Moskov. Gos. Univ. Uč. Zap. **155**, Mat. **5** (1952), 3–53; MR **17**, 992] that compact mappings cannot lower Borel class. Now the author shows that if the class is infinite, compact mappings preserve it. Further details are given. Also (Theorem 3): Among metric spaces, closed mappings take absolute G_δ's to absolute G_δ's. Theorem 4: Let f be a closed mapping of a separable metric space X upon a space Y which is nowhere first category; then there is $H \subset X$ such that $f|H$ is a homeomorphism upon a dense G_δ in Y.

J. R. Isbell (Seattle, Wash.)

9958:

Aquaro, Giovanni. Un criterio di compattezza per insiemi di applicazioni non continue. Rend. Accad. Sci. Fis. Mat. Napoli (4) **26** (1959), 375–382. (English summary)

This paper establishes some criteria for compactness (relative to the topology of uniform convergence) of sets of functions defined on a set E (which is not necessarily topologized) and with values in a Hausdorff uniform space E'. Such a condition is that the set is pointwise relatively compact and that for every entourage V' for E' there is a finite dissection $\{A_k\}$ of E such that if $x, y \in A_k$, then $(f(x), f(y)) \in V'$ for all f in the set.

R. G. Bartle (Urbana, Ill.)

9959:

de Groot, J. Groups represented by homeomorphism groups. I. Math. Ann. **138** (1959), 80–102.

When is a group isomorphic to the group of all autohomeomorphisms of some reasonable topological space or to the automorphism group of some algebraic structure of specified type? The author proves that every group is isomorphic to a group $A(M)$ of all autohomeomorphisms (indeed, all isometries) of a connected, locally connected, complete metric space M of any preassigned dimension $\geqq 1$, and as a corollary, using the theory of rings of continuous functions, that every group is isomorphic to the automorphism group of a commutative ring with unit. The construction of the space M is effected by first representing the given group as the automorphism group of a directed graph (Cayley graph), then replacing each edge of the graph by a topologically rigid metric space of the desired dimension, and finally introducing a metric in the whole space.

The groups $A(M)$ for zero-dimensional spaces M are more special, and various properties of these are investigated. If N is either a denumerable metric space or an infinite locally compact zero-dimensional separable metric space, then $A(N)$ is equivalent to the group $S_{\aleph_0}$ of all permutations of a denumerable set in the sense that each group is isomorphic to a subgroup of the other. Thus the automorphism group of any denumerable boolean ring is also equivalent to $S_{\aleph_0}$. If B is an atomless boolean ring of power $\geqq C$ (the power of the continuum), then its automorphism group is either trivial or else it contains infinitely many elements of order two. (This contradicts some results of Rieger, Fund. Math. **38** (1951), 209–216 [MR **14**, 238]; p. 214.)

In studying rigid spaces of several types the author proves that there is a family $\{F_\alpha\}$ of 2^C zero-dimensional subsets of the real line, none of which can be mapped continuously and non-trivially into itself or another F_α; also, there is a family of 2^C one-dimensional, connected and locally connected plane sets with the same property.

Eva Kallin (Berkeley, Calif.)

9960:

Nikaidô, Hukukane. Coincidence and some systems of inequalities. J. Math. Soc. Japan **11** (1959), 354–373.

The author states a very general theorem on the existence of solutions to systems of inequalities, with a proof based on Vietoris cohomology theory. A function $f: S \to T$ is termed a Vietoris map if $f^{-1}(q)$ is acyclic for all $q \in T$. The principal theorem proved is the following. (i) Let S, X, and Y_i $(i=1, \cdots, m)$ be Hausdorff spaces. Furthermore, S is compact, and each Y_i is a compact convex set in which the convex linear combination of finitely many points depends continuously on its coefficients. (ii) For each $i=1, \cdots, m$, there is a real-valued function $h_i(x, y_i)$ on $X \times Y_i$ which is continuous on X for every fixed $y_i \in Y_i$ and quasi-convex on Y_i for every fixed $x \in X$. (iii) There is a Vietoris map $f: S \to Y = Y_1 \times \cdots \times Y_m$. (iv) $\phi: S \to 2^X$ is an upper semi-continuous map whose image sets $\phi(p)$ are compact and acyclic. (v) $g(p)$ is a lower semi-continuous real-valued function on S. (vi) $\sum_{i=1}^m h_i(x, f_i(p)) \geqq g(p)$ for any p, x such that $x \in \phi(p)$, where $f_i(p)$ stands for the projection of $f(p)$ on Y_i. If conditions (i)–(vi) are fulfilled, then at some point $\bar{p} \in S$, the corresponding image $\phi(\bar{p})$ contains a point $\bar{x}$ such that $\sum_{i=1}^m h_i(\bar{x}, y_i) \geqq g(\bar{p})$ for all $y_i \in Y_i$ $(i=1, \cdots, m)$.

This theorem can be applied to yield known theorems about economic equilibrium and equilibrium points for non-cooperative games (including a minimax theorem as a special case). The economic equilibrium theorem assumes finitely many commodities and a set-valued excess supply function of prices, upper semi-continuous with each image set compact and acyclic (usually convex in application). A novel generalization of the economic equilibrium theorem to infinitely many commodities is stated; it is proved by approximations based on the above theorem.

K. J. Arrow (Stanford, Calif.)

9961:

Gary, John. The topological structure of trajectories. Michigan Math. J. **7** (1960), 225–227.

Let V_n be a connected differentiable manifold and let $X(x, t)$, $x \in V_n$, $t \in E_1$, define a field of tangent vectors on $V_n \times E_1$. The trajectories considered are the curves $(x(t), t) \subset V_n \times E_1$, where $x(t)$ is a solution of the equations $dx/dt = X(x, t)$. A section on $V_n \times E_1$ is a manifold V_n' together with a homeomorphism of V_n' into $V_n \times E_1$ with the property that $f(V_n')$ meets each trajectory exactly once. Let V be the space secured by letting each trajectory of $V_n \times E_1$ be a point. If V is a Hausdorff space, then V is an n-dimensional manifold and if $X(x, t)$ and V_n are of class C^n, then V is also. If a differential system on $V_n \times E_1$ admits a section by V_n', then there exists a homeomorphism of $V_n' \times E_1$ onto $V_n \times E_1$ which maps the lines $y \times E_1$, where $y \in V_n'$, onto trajectories. If $X(x, t)$ is of class C', the differential system admits a section if, and only if, the identification space V is a Hausdorff space, and if $f: V_n' \to V_n \times E_1$ is a section for the system, then V_n' is homeomorphic to V.

W. R. Utz (Columbia, Mo.)

9962:

Nagata, Jun-iti. On rings of continuous functions and the dimension of metric spaces. Proc. Japan Acad. **36** (1960), 49–52.

It is well known [M. Katětov, Časopis Pěst. Mat. Fys. **75** (1950), 1–16, 79–87; MR **12**, 119] that if R is a compact metric space, then dim R is equal to the smallest cardinal of an analytic base for $C^*(R)$, the ring of bounded continuous functions [cf. also L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand, Princeton, N.J., 1960; MR **22** #6994]. The author introduces the notion of an analytic base of a subring C' of an "analytic ring" C and proves that, for a locally compact metric R, dim R is equal to the smallest cardinal of an analytic base of $U(R)$ in $C^*(R)$, $U(R)$ being the ring of all bounded uniformly continuous functions.

M. Katětov (Prague)

9963:

Nagata, Jun-iti. Two remarks on dimension theory for metric spaces. Proc. Japan Acad. **36** (1960), 53–54.

The following theorems are proved for a metric space R: (1) R is countable-dimensional if and only if for any closed F_i and open U_i with $F_i \subset U_i$, $i=1, 2, \cdots$, there exist open V_i such that $F_i \subset V_i \subset U_i$ and $\{\overline{V}_i - V_i\}$ is point-finite; (2) if $R = \bigcup_{\alpha<\tau} F_\alpha$, dim $F_\alpha \leqq n$, $\bigcup_{\alpha<\beta} F_\alpha$ closed for every $\beta < \tau$, then dim $R \leqq n$.

M. Katětov (Prague)

ALGEBRAIC TOPOLOGY

See also 9960.

9964:

Cairns, Stewart Scott. ★**Introductory topology.** The Ronald Press Co., New York, 1961. ix+244 pp. $8.75.

"This book is the culmination of repeatedly revised sets of class notes used by the author in teaching introductory topology courses. Its purpose is to progress as far as practicable into the fundamental concepts and the principal results of homology theory, both in their combinatorial development and in their application to topological spaces."

The chapter headings are: Ch. 1, Illustrative geometric examples; Ch. 2, Topological classification of surfaces; Ch. 3, Introduction to set-theoretic topology; Ch. 4, Complexes; Ch. 5, Homology and cohomology groups; Ch. 6, Topological invariance of homology properties; Ch. 7, Manifolds; Ch. 8, The fundamental group, covering surfaces; Appendix, Group-theoretic background. There is also a bibliography, an index of symbols, and a general index.

This list indicates that the scope of the book is highly comparable with that of Seifert and Threlfall, *Lehrbuch der Topologie* [Teubner, Leipzig, 1934], and unilingual students in English-speaking countries meeting topological notions for the first time may well be happy to turn to the book under review rather than to the now classical German text. Certainly they will be rewarded by being presented with a treatment of elementary homology theory in which the motivation is always strongly emphasized and in which the relationship of combinatorial to topological concepts is made extremely clear. The first three chapters make very stimulating reading and the drawings, especially in Chapter 2, are of an astonishingly high standard.

However, the reviewer was less happy about the more algebraic aspects of the treatment. The author rarely abstracted the appropriate algebraic object from the topological or combinatorial context and considered it as a thing in itself; it is significant that 'chain complex' and 'chain map' find no place in the index. The algebraic notation was highly non-standard (capital letters being used for chains, for example, and Φ for the fundamental group; and ϕ stood at times for the empty set, the zero element of a group, and the zero group); all this may well lead to difficulties for the student who wishes to go further. The appendix lacked the clarity and elegance of the author's treatment of geometry. Thus, the notion of the reduced form of an element seemed to the reviewer to be formulated in such a way that, by the author's criterion, every group is free; on p. 220 (C), the homomorphism h is tacitly supposed to be onto; and the definition of direct product on p. 221 (actually the internal direct product) was made in terms of three conditions of which the second was implied by the third and the third contained the superfluous phrase 'except 0'. Matrices figured prominently—but here questions of taste are involved; however, an issue of principle arises in connection with the author's treatment of cohomology. It is, of course, thoroughly appropriate to introduce cohomology at this level, and the author makes some use of it in formulating the duality theorems of Chapter 7. However, he identifies chains with cochains and thus loses most of the algebraic flavour of cohomology. Indeed, considerations of naturality find no place in the entire treatment and the notions 'covariant', 'contravariant' are applied only to components of a vector. The reviewer believes that it is important at an early stage to bring out the role of the group structure (or ring structure) in homology theory and not to concentrate undue attention on Betti numbers and torsion coefficients.

The reviewer would not wish to end this review of a book with so many excellent features on a critical note. As an introductory text in topology—especially as a course book accompanying a stimulating series of lectures—it should be recommended. It will clarify in the student's mind the sort of questions the topologist asks and show how he sets about trying to answer them; but it may not adequately represent the way in which the algebraic topologist formulates his answers.

P. J. Hilton (Birmingham)

9965:

Mukohda, Shunji. On a theorem of Toda in the Steenrod algebra. Mem. Fac. Sci. Kyushu Univ. Ser. A **14** (1960), 85–97.

"Let S^* denote the Steenrod algebra corresponding to an odd prime p, and define an operation α_* [resp. α^*]: $S^* \to S^*$ by $\alpha_*(\beta) = \alpha\beta$ [resp. $\alpha^*(\beta) = \beta\alpha$] for $\alpha, \beta \in S^*$. Then the kernel of $P_*^{p^t}: S^* \to S^*/M_t$ has been determined by H. Toda in a recent paper [Mem. Coll. Sci. Univ. Kyoto Ser. A **31** (1958), 129–142; MR **21** #4420a], where M_t designates a certain ideal of S^*. This theorem was used to determine the triviality or non-triviality of the mod p Hopf invariant. The proof of the theorem, however, seems to be incomplete. The aim of the present note is to give another proof using Milnor's basis instead of the usual one in S^*." (From the author's introduction.)

Needless to say, the entire paper consists of elaborate calculations with the Steenrod algebra S^*.

W. S. Massey (New Haven, Conn.)

9966:

Okamoto, Sigeru. On reduced product complexes. Sci. Rep. Tokyo Kyoiku Daigaku. Sect. A **6**, 229–237 (1959).

For every chain complex C the tensor algebra RC is defined as $RC = \mathbf{Z} + C + C \otimes C + \cdots = \sum_{i=0}^{\infty} \otimes^i C$. Similarly the tensor algebra of an FD-module K is the FD-algebra $F^+K = P + K + K^2 + \cdots = \sum_{i=0}^{\infty} K^i$ (this construction is the algebraic version of James' reduced product complex). The author proves that F^+K and RK are homotopy equivalent. This is an immediate consequence of the Eilenberg-Zilber theorem as formulated by Eilenberg and MacLane [Ann. of Math. (2) **60** (1954), 49–139; MR **16**, 391; p. 51].

A. Dold (New York)

9967:

Koh, S. S. Note on the homotopy properties of the components of the mapping space X^{S^p}. Proc. Amer. Math. Soc. **11** (1960), 896–904.

Following G. W. Whitehead [Ann. of Math. (2) **47** (1946), 460–475; MR 8, 50] and S. T. Hu [Indag. Math. 8 (1946), 623–629; MR 8, 481] the author studies the function-space $G^p(X)$ of maps from S^p to X. The component corresponding to an element $\alpha \in \pi_p(X)$ is written $G_\alpha^p(X)$; the author is particularly concerned to determine whether the different components $G_\alpha^p(X)$ have the same

homotopy type. This is true, for example, if X is an arcwise-connected H-space. After such generalities, the author specialises to the case $X=S^r$ and finally to the case $p=r$, so that $\alpha \in \pi_r(S^r)$. He then proves the following. (i) For even $r \leq 14$, the components $G_\alpha{}^r$ and $G_\beta{}^r$ have the same homotopy type if and only if $\alpha = \pm\beta$. (ii) For odd $r \neq 1, 3, 7$, the components $G_\alpha{}^r$ and $G_\beta{}^r$ can have the same homotopy type only if $\deg \alpha - \deg \beta$ is even. (The cases $r=1, 3, 7$ are settled by a remark given above.) This result is accompanied by some explicit values of homotopy groups $\pi_q(G_\alpha{}^p)$; among these, Lemma 3.14 perpetuates a well-known error. If one tries to avoid such details, it is fairly easy to obtain result (i) above for all even r.

J. F. Adams (Cambridge, England)

9968:

Reinhart, Bruce L. Simple curves on compact surfaces. Proc. Nat. Acad. Sci. U.S.A. **46** (1960), 1242–1243.

Suppose M is a closed 2-manifold of genus $p \geq 2$. The author considers the problem, given an element γ of $\pi_1(M, x')$: when is there a simple closed curve on M representing γ. An algorithm is given to decide if this is possible for a given γ. For this the author studies the hyperbolic plane covering M.

S. Smale (Berkeley, Calif.)

9969:

Miyazaki, Hiroshi. Correction: On realizations of some Whitehead products. Tôhoku Math. J. (2) **12** (1960), 480.

This correction to same J. **12** (1960), 1–30 [MR **22** #5971] acknowledges the falsity of Lemma 8 and re-proves Theorem 4.

9970:

Papakyriakopoulos, C. D. The theory of three-dimensional manifolds since 1950. Proc. Internat. Congress Math. 1958, pp. 433–440. Cambridge Univ. Press, New York, 1960.

This is a summary of important theorems on 3-manifolds between 1950 and 1958. The areas in which major advances are reported, with the corresponding authors, are as follows: (1) semi-linear homeomorphisms (W. Graeub); (2) triangulability and the Hauptvermutung (E. E. Moise, R. H. Bing, J. Munkres); (3) separation properties (E. E. Moise); (4) tameness properties (R. H. Bing, O. G. Harrold, H. C. Griffith, E. E. Posey); (5) polyhedral and semi-linear approximations (R. H. Bing, D. E. Sanderson); (6) Dehn's Lemma (C. D. Papakyriakopoulos, T. Homma, J. H. C. Whitehead and A. S. Shapiro); (7) non-contractible 2-spheres in 3-manifolds (C. D. Papakyriakopoulos, J. H. C. Whitehead); (8) asphericity (C. D. Papakyriakopoulos, R. J. Aumann); (9) the classification problem for closed 3-manifolds (J. Milnor); (10) the Poincaré conjecture (R. H. Bing, C. D. Papakyriakopoulos).

S. S. Cairns (Urbana, Ill.)

9971:

Brody, E. J. On infinitely generated modules. Quart. J. Math. Oxford Ser. (2) **11** (1960), 141–150.

Let M be a manifold with finitely presented fundamental group G, and with a natural homomorphism θ from the group ring of G over the integers into a unique factorization domain P. Under θ a relation module for G provides a finitely generated P-module $\mathfrak{M}$, whose elementary divisors yield topological invariants of M. (See R. H. Fox [Ann. of Math. (2) **57** (1953), 547–560; **59** (1954), 196–210; MR **14**, 843; **15**, 931] and G. Torres [ibid. (2) **57** (1953), 57–89; MR **14**, 574].) These ideas are extended to modules $\mathfrak{M}$ that are not finitely generated, which arise in particular in connection with wild knots.

Each elementary ideal $\Delta_k(\mathfrak{M}')$, for $\mathfrak{M}'$ a submodule or quotient module of a finitely generated $\mathfrak{M}$, includes $\Delta_k(\mathfrak{M})$. For general $\mathfrak{M}$, the family of $\Delta_k(\mathfrak{M}')$ for finitely generated submodules $\mathfrak{M}'$, as neighborhoods of 0, define a 'divisor topology' $\nabla_k(\mathfrak{M})$ on P. The $\nabla_k(\mathfrak{M}/\mathfrak{M}')$ define a 'residual topology' $\Gamma_k(\mathfrak{M})$ on P. The 'nullity' of $\mathfrak{M}$ is the supremum of k such that $\Delta_k(\mathfrak{M}') \neq 0$ for some finitely generated submodule $\mathfrak{M}'$. These concepts are shown to have the properties one would naturally desire, and some discussion of computation is given. In the important case that the Betti group of (the commutator quotient of) G is free abelian, results are obtained related to those of Torres on the Alexander polynomial. It is shown that the nullity of a knot group is positive, and is 1 unless the knot is wild, while examples are given of wild knots with nullities 1, 2, and 3. A geometrical interpretation of the nullity of a knot is conjectured.

R. C. Lyndon (London)

9972:

Wilder, R. L. A certain class of topological properties. Bull. Amer. Math. Soc. **66** (1960), 205–239.

In his long and exhaustive study of the topology of generalised manifolds, the author has invented or developed many concepts expressed in terms of the injection of one homology group in another. The applications of these concepts were mainly "local", but the methods frequently apply also to "medial" situations, neither purely local nor global. Such "medial" concepts are studied fairly systematically in the present paper, as for example the properties "$(P, Q)_r$": a subset M of a topological space X has property $(P, Q)_r$ [resp. $(P, Q, \sim)_r$], if for every pair P, Q of open subsets of X with $P \supseteq Q$ and $\bar{Q}$ compact, the image of i [resp. $i\delta$] in the sequence

$$h_{r+1}(M, M \cap Q) \xrightarrow{\delta} h_r(M \cap Q) \xrightarrow{i} h_r(M \cap P)$$

is finitely generated. (h denotes homology with compact carriers.) Cohomology properties are defined similarly. Many theorems relating these properties are proved, of which a sample is the "Fourth Duality Theorem": If F is a closed subset of an n-gm M, then a necessary and sufficient condition that F have property $(P, Q, \sim)_r$ for some $r \in \{0, \cdots, n-1\}$ is that $M-F$ have property $(P, Q, \sim)_{n-r-2}$. Property $(P, Q)_r$ is closely related to local connectivity; for example, if F above is a continuum, then F is lc^r ($r \leq n-2$) if and only if $M-F$ has properties $(P, Q)_s$ ($n-r-1 \leq s \leq n-2$) and $(P, Q, \sim)_{n-r-2}$. Other theorems are proved concerning frontiers of regions, e.g., a generalised Torhorst theorem. The paper concludes with a study of the effect of monotone mappings on these medial properties.

H. B. Griffiths (Birmingham)

9973:

Araki, S.; James, I. M.; Thomas, Emery. Homotopy-abelian Lie groups. Bull. Amer. Math. Soc. **66** (1960), 324–326.

This paper extends the results and methods of James

and Thomas, Proc. Nat. Acad. Sci. U.S.A. **45** (1959), 737–740 [MR **21** #5197]. The main result is the following theorem: A compact connected Lie group is homotopy-abelian only if it is abelian.

R. Bott (Cambridge, Mass.)

9974:

Curtis, M. L.; Zeeman, E. C. On the polyhedral Schoenflies theorem. Proc. Amer. Math. Soc. **11** (1960), 888–889.

The authors construct a certain 3-manifold M^3 which has the homology of S^3 but is not simply connected, a "Poincaré manifold". It is an open question, if the double suspension of M^3 is homeomorphic to S^5. The authors show that if the answer to this question is affirmative, then certain versions of the polyhedral Schoenflies problem fail.

S. Smale (Berkeley, Calif.)

9975:

Milnor, J. On the cobordism ring Ω^* and a complex analogue. I. Amer. J. Math. **82** (1960), 505–521.

In this paper the author proves that the cobordism groups Ω^i defined by Thom [Comment. Math. Helv. **28** (1954), 17–86; MR **15**, 890] have no odd torsion. Thom proved that the cobordism groups Ω_i are isomorphic to the stable homotopy groups $\pi_{i+n}(M(SO_n))$ of certain "Thom spaces" $M(SO_n)$, and it is these stable homotopy groups which the author handles. Analogous Thom spaces $M(U_n)$ can be defined starting from the unitary group; the author's methods apply equally well to the stable homotopy groups $\pi_{i+2n}(M(U_n))$; we are promised that in part II of this paper these groups will be interpreted as "complex cobordism groups". For the moment he shows that they have no torsion and determines their structure.

The author handles his stable homotopy groups by using a spectral sequence due to the reviewer [ibid. **32** (1958), 180–214; Bull. Soc. Math. France **87** (1959), 277–280; MR **20** #2711; **22** #8500]. If X and Y are two finite CW-complexes, then this spectral sequence serves to relate the stable track groups $\{X, Y\}_n$ to the Ext groups $\mathrm{Ext}_A^{s,t}(H^*(Y; Z_p), H^*(X; Z_p))$. Here A denotes the mod p Steenrod algebra, and the cohomology groups H^* are regarded as modules over A. More generally, it is possible to replace Y by an object in a suitable category of stable objects. This is convenient here, because it enables one to replace the sequence of Thom spaces $M(SO_n)$ by a single "stable Thom object" $\mathbf{M}(SO)$. The cobordism group Ω^n is then isomorphic to the stable track group $\{S^0, \mathbf{M}(SO)\}_n$, and in principle this can be computed by the spectral sequence, taking X to be the sphere S^0.

In an earlier draft of this paper this computation was actually carried through. However, conclusions about the absence of torsion follow more easily from the following. Theorem 1: If $H^*(\mathbf{Y}; Z_p)$ is a free $A/(Q_0)$-module with even-dimensional generators, and if $C_*(\mathbf{Y}; Z)$ satisfies a finiteness condition, then the stable groups $\{S^0, \mathbf{Y}\}_n$ contain no p-torsion. (Q_0) denotes the two-sided ideal in A generated by the Bockstein boundary β_p. This theorem is proved very elegantly by taking X to consist of a circle with a 2-cell attached by a map of degree d. In this case everything in the spectral sequence is even-dimensional, and consequently $\{X, \mathbf{Y}\}_m$ is zero for m odd. On the other hand, if some $\{S^0, \mathbf{Y}\}_m$ contained p-torsion, then $\{X, \mathbf{Y}\}_n$ would have to be non-trivial for two consecutive values of n; this follows from the universal coefficient theorem for expressing $\{X, \mathbf{Y}\}_m$ in terms of $\{S^0, \mathbf{Y}\}_n$.

It remains, of course, to show that $H^*(\mathbf{M}(SO); Z_p)$ and $H^*(\mathbf{M}(U); Z_p)$ are free $A/(Q_0)$-modules with even-dimensional generators. This is done by adapting the argument which Thom used in the case $p=2$ to show that $H^*(\mathbf{M}(O); Z_2)$ is a free A-module.

The paper is arranged as follows. After an introduction, § 1 contains necessary lemmas concerning the structure of the Steenrod algebra A, of $A/(Q_0)$, and of modules over these. These lemmas are based on the author's earlier work in Ann. of Math. (2) **67** (1958), 150–171 [MR **20** #6092]. § 2 is about stable objects, the spectral sequence, and theorem 1. § 3 treats the particular stable object $\mathbf{M}(U)$, its cohomology and stable homotopy groups. § 4 indicates how $\mathbf{M}(SO)$ can be treated by trivial amendments of the details in § 3. The paper concludes with applications and suggestions for further research.

One regrets that this significant paper could not have appeared earlier, in some form; but one can only applaud the elegance and clarity of the final version.

J. F. Adams (Cambridge, England)

9976:

Fox, Ralph H. The homology characters of the cyclic coverings of the knots of genus one. Ann. of Math. (2) **71** (1960), 187–196.

Let $\mathfrak{M}_g$ be the gth cyclic covering of the spherical 3-space $\mathfrak{M}_1$ branched about a knot k of genus one in it. On the basis of recursion formulae defined by A. Plans [cf. Rev. Acad. Ci. Madrid **47** (1953), 161–193; MR **15**, 147] the author obtains (Theorem 1) formulae which permit easy calculation of the torsion numbers τ_1, τ_2 and the invariants $\chi_1(p)$, $\chi_2(p)$ of self-linking [cf. R. C. Blanchfield and R. H. Fox, same Ann. (2) **53** (1951), 556–564; MR **12**, 730] of $\mathfrak{M}_g$. By means of Theorem 1 the author proves further that $H_1(\mathfrak{M}_g)$ is finite in all cases except $\gamma=1$, $g\equiv 0 \pmod 6$, where γ denotes the absolute term of the Alexander polynomial $\Delta(t)=\gamma t^2+(1-2\gamma)t+\gamma$ of k; that $\mathfrak{M}_g$ is a Poincaré space if and only if either $\gamma=0$ or $\gamma=1$ and $g\equiv\pm 1 \pmod 6$; that if $g|g'$ then the torsion numbers of $\mathfrak{M}_g$ divide the corresponding numbers of $\mathfrak{M}_{g'}$; and finally that if $(4\gamma-1)/w^2$ has a prime divisor of the form $4N+3$, where w (=g.c.d.$(v_{11}, v_{12}+v_{21}, v_{22})$) denotes the positive generator of the elementary ideal $\mathfrak{E}_2(t)$ for $t=-1$ of the Alexander matrix of the Seifert form, then $\mathfrak{M}_g$ is asymmetric for every even g, and κ (the knot type of k) is not amphicheiral. As the knots of genus one constitute a reasonably large class, the results obtained are considered by the author to be indicative of possible general results.

H. Terasaka (Osaka)

DIFFERENTIAL TOPOLOGY

See also 9961, 9973.

9977:

Scheja, Günter. Prolongement de Riemann concernant les classes de cohomologie. C. R. Acad. Sci. Paris **251** (1960), 2863–2864.

Let X be a complex manifold and $\mathfrak{O}$ be its structural

sheaf. For a coherent analytic sheaf $\mathfrak{F}$ on X, the notion of the homological dimension $\mathrm{hd}(\mathfrak{F}_x)$ of the stalk $\mathfrak{F}_x$ is defined by the least number d for which there exists an exact sequence

$$0 \to \mathfrak{O}_x^{q_d} \to \cdots \to \mathfrak{O}_x^{q_0} \to \mathfrak{F}_x \to 0$$

of $\mathfrak{O}_x$-homomorphisms. Now, the author proves the following theorem. If A is a set holomorphically thin of codimension $\geq m_x$, the canonical restriction

$$H^p(X, \mathfrak{F}) \to H^p(X-A, \mathfrak{F})$$

is bijective for $p \leq m_x - \mathrm{hd}(\mathfrak{F}_x) - 2$, and injective for $p \leq m_x - \mathrm{hd}(\mathfrak{F}_x) - 1$.

This theorem implies various theorems on removable singularities. For example, setting $\mathfrak{F} = \mathfrak{O}$ ($\mathrm{hd}(\mathfrak{O}_x) = 0$), $m_x = 2$, $p = 0$, we have a classical theorem on the prolongation of Riemann, saying that a set holomorphically thin of codimension 2 is a removable singularity for holomorphic functions. Similarly, for $\mathfrak{F} = \mathfrak{O}$, $m_x = 3$, $p = 1$, it gives a theorem on the prolongation of the Cousin first distribution due to W. Rothstein. Finally, a theorem on normality of an analytic set due to Rothstein-Oka is also implied in the above result. *S. Hitotumatu* (Tokyo)

9978:

Papy, Georges. Espaces différentiels. Colloque Géom. Diff. Globale (Bruxelles, 1958), pp. 163–180. Centre Belge Rech. Math., Louvain, 1959.

Given a space E and a family of real-valued functions $X: E \to R$, a differentiable structure is defined and the exterior differential algebra developed. Through the use of supports one constructs a complex in the sense of Leray.

The functions in the family X are assumed to separate the space E and are called the generators. E is given the weakest topology relative to which the generators are continuous. A function $f: E \to R$ is called differentiable if there exist continuous functions $a_i: E \times E \to R$ ($i = 1, \cdots, n$) such that for every $(p, q) \in E \times E$

$$f(q) - f(p) = \sum_i a_i(p, q)[x_i(q) - x_i(p)],$$

where $x_i \in X$.

The difference of a function δh is defined by $\delta h(p, q) = h(q) - h(p)$, and its differential by $dh = \lim \circ \delta h$. The operator lim is a homomorphism of the bi-module of continuous functions: $E \times E \to R$ whose kernel is the set of "infinitesimals of higher order". Thus one can define Pfaffian forms of class C^1 on a C^1 structure, which is not possible in the classical case. The exterior differential algebra of E is then developed by use of the graded ring of Alexander-Kolmogoroff.

C. B. Allendoerfer (Seattle, Wash.)

9979:

Huebsch, William; Morse, Marston. A singularity in the Schoenflies extension. Proc. Nat. Acad. Sci. U.S.A. **46** (1960), 1100–1102.

Mazur has proved that a differentiable imbedding $\phi: S^{n-1} \to E^n$ can be extended to a homeomorphism of the unit ball. The second named author showed the extension could be chosen as a diffeomorphism except at one point. Here it is shown that this point may be prescribed.

S. Smale (Berkeley, Calif.)

9980:

Hirsch, Morris W. Immersions of manifolds. Trans. Amer. Math. Soc. **93** (1959), 242–276.

The main result of this paper is the classification of regular homotopy classes of immersions of a closed differentiable manifold M into a differentiable manifold N in the case where $\dim M < \dim N$. The classification theorem (§ 5) reads as follows: Let $\dim M = k$, and let $T_k(M)$ and $T_k(N)$ be the spaces of tangent k-frames over M and N respectively. Every immersion $f: M \to N$ induces a $\mathrm{GL}(k)$-equivariant map $f_*: T_k(M) \to T_k(N)$, and a regular homotopy between two immersions $f, g: M \to N$ induces a $\mathrm{GL}(k)$-equivariant homotopy between the induced maps f_*, g_*. Claim: The correspondence between regular homotopy classes of immersion $M \to N$ and $\mathrm{GL}(k)$-equivariant homotopy classes of $\mathrm{GL}(k)$-equivariant maps $T_k(M) \to T_k(N)$ is one-to-one provided $\dim M < \dim N$. [The surjectivity was conjectured by Ch. Ehresmann in C. R. Acad. Sci. Paris **226** (1948), 1879–1880; MR **10**, 56.] By use of a general remark on fibre bundles (Theorem 5.1), the classification of $\mathrm{GL}(k)$-equivariant homotopy classes of $\mathrm{GL}(k)$-equivariant maps $T_k(M) \to T_k(N)$ is seen to be equivalent to classifying (ordinary) homotopy classes of cross-sections in a certain bundle with base space M^k and fibre $T_k(N)$. For instance, if $M = S^k$ (and $k < \dim N$), the regular homotopy classes of immersions $S^k \to N$ are in one-to-one correspondence with the elements of $\pi_k(T_k(N))$.

The author's method of proof is by using the result in the special case $M = S^k$, $N =$ Euclidean space, as proven by S. Smale [Ann. of Math. (2) **69** (1959), 327–344; MR **21** #3862], and building up an "obstruction theory". A basic notion for this purpose is the concept of an M-immersion into the manifold N of a subset $K \subset M$. A bundle map $f': T(M|K) \to T(N)$, where $T(M|K)$ is the restriction to K of the tangent bundle of M, is said to be an M-immersion of K if it is the derivative on K of an immersion $f: U \to N$, where U is some neighborhood of K in M. An M-immersion of M is just an immersion of M. An immersion $f: M \to N$ with a given $\mathrm{GL}(k)$-equivariant homotopy class of induced map $f_*: T_k(M) \to T_k(N)$ is constructed by stepwise extension of an M-immersion of the i-skeleton K^i of a smooth triangulation of M ($i = 0, 1, \cdots, k$). A similar process applies to the construction of a regular homotopy between immersions $f, g: M \to N$ with equivariantly homotopic induced maps $f_*, g_*: T_k(M) \to T_k(N)$.

As a corollary of the methods of proof, it is shown (Theorem 6.4) that the existence of a field of normal r-frames for an immersion $f: M \to E^{n+r}$ into Euclidean space is a necessary and sufficient condition for the compressibility of the immersion into E^n. Various applications are then derived from this theorem, using the characteristic classes.

The paper can be read without going back to Smale's paper (quoted above) in spite of the fact that essential use is made of Smale's result. The theorems of Smale which are needed are stated explicitly (and slightly generalized) by the author. The exposition is very detailed.

M. A. Kervaire (Paris)

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